An Introduction of Topological Orders

Xiao-Gang Wen
Dept. of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
http://dao.mit.edu/˜wen

Abstract

One of most fundamental issues about matter is to understand states of matter and the associated internal orders. In this article, we examine the orders in states of matter in depth and present a new kind of order – topological order. Using fractional quantum Hall states as examples, a simple intuitive picture, the unique properties, and the experimental measurements of topological order are discussed. To gain a deeper understanding of topological order, a concept – quantum order – is introduced to describe a new kind of orders that generally appear in quantum states at zero temperature. Quantum orders that characterize universality classes of quantum states (described by *complex* ground state wave-functions) is much richer then classical orders that characterize universality classes of finite temperature classical states (described by *positive* probability distribution functions). Topological order is a special case of quantum order. Topological orders (and quantum orders) extend and deepen our previous understanding of orders in states of matter, and guide us to discover new states of matter.

1 States of matter and internal orders

Matter can have many different states, such as gas, liquid, and solid. The state of matter is one of the most important properties of matter. Scientists find matter can have much more different states than just gas, liquid, and solid. Even solids and liquids can appear in many different forms and states. In this article, we are going to study what is behind those different states of matter and what distinguish different states of matters.

Different states of matter are found to be distinguished by their internal structures. Those internal structures are called the orders. At high enough temperatures, matters are in a form of gas. All atoms in a gas move randomly, independent of other particles. Thus gas is a very disordered and uncorrelated state. However, as temperature is lowered, the motion of atoms becomes more and more correlated (that is the motion of one atom depend on the motions of other atoms). Eventually the atoms form a very regular pattern and a crystal order is developed. In fact atoms can form many different crystal orders, depending on the interaction between atoms, the temperatures and the pressures. In addition to the crystals, physicists discovered many other states of matter in last century, such as superfluids, ferro and antiferromagnets, and liquid crystals that appear in every calculator and electronic watches. All those states of matter have different internal structures or orders.

With so many different states of matter, a general theory is needed to gain a deeper understanding of states of matter and the associated internal orders. The key step in developing the general theory is the realization that all the orders are associated with symmetries (or rather, the breaking of symmetries). For example when a gas changes into a crystal, its symmetry changes. A gas remains the same under a translation by any distance, while a crystal remains the same only under a translation by the lattice constant of the crystal. Thus the development of a crystal order reduces the continuous translation symmetry of a gas to a discrete translation symmetry of a crystal. Based on the relation between orders and symmetries, Landau developed a general theory of orders and their transitions. Landau's theory is so successful and one starts to have a feeling that we understand, at in principle, all kinds of orders that matter can have.

2 Fractional quantum Hall liquids

However, nature never stops to surprise us. With advances of semiconductor technology, physicists learned how to confine electrons on an interface of two different semiconductors, and hence making a two dimensional electron gas (2DEG). In 1982, Tsui, Stormer, and Gossard put a 2DEG under strong magnetic fields (~ 30 Tesla) in the Magnet Lab at MIT and cool it to very low temperatures ($\sim 1K^\circ$).[2] They found that the 2DEG forms a new kind of state – Fractional Quantum Hall (FQH) state. Since the temperatures are low and interaction between electrons are strong, the FQH state is a strongly correlated state. However such a strongly correlated state is not a crystal as people originally expected. It turns out that the strong quantum fluctuations of electrons due to their very light mass prevent the formation of a crystal. Thus the FQH state is a special kind of liquid called quantum liquid. (A crystal can be melted in two ways: (a) by thermal fluctuations as we raise temperatures which leads to an ordinary classical liquid; (b) by quantum fluctuations as we reduce the mass of the particles which leads to a quantum liquid.)

Physicists soon discovered many amazing properties of quantum Hall liquids. A quantum Hall liquid is more "rigid" than a solid (a crystal), in the sense that a quantum Hall liquid cannot be compressed. Thus a quantum Hall liquid has a fixed and well-defined density. More magic shows up when we measure the electron density in terms of filling factor ν. The filling factor is defined as a ratio of the electron density n and the density of the flux quanta of the applied magnetic field B

$$\nu \equiv \frac{nhc}{eB} = \frac{\text{density of electrons}}{\text{density of magnetic flux quanta}}$$

Physicists found that all discovered quantum Hall states have such densities that the filling factors are exactly given by some rational numbers, such as $\nu = 1, 1/3, 2/3, 2/5, \ldots$. quantum Hall states with simple rational filling factors (such as $\nu = 1, 1/3, 2/3, \ldots$) are more stable and easier to observe, while quantum Hall states with complex rational filling factors (say $\nu = 4/9, 2/7, \ldots$) are less stable and harder to observe (*ie* they appear only in cleaner samples and lower temperatures).

The quantum Hall states with integer filling factors are called the integral quantum Hall (IQH) states, and the ones with fractional filling factors are called FQH states. The IQH quantum Hall states, first discovered by van Klitzing in 1980,[1] can be easily understood from the Landau level structure in the strong magnetic field. However, the understanding of FQH states requires a whole new theory. The internal structure of FQH state is so new that previous many-body theory developed for other electron systems, such as metals and semiconductors, simply do not apply to FQH states. It was Laughlin[3] who approached FQH states from a completely new angle that starts our theoretical understanding of FQH effects.

The history of quantum Hall effect is very interesting. Since the discovery of IQH and FQH effects, there are a few times when people feel that they understand FQH systems and it is time to leave the subject. But new experimental discoveries and new theoretical insights re-attract people's attention and generate new cycles of intensive studies. Each cycle leads to a new fresh point of view and much deeper understanding of FQH systems, as if a whole new world was discovered. 18 years after its discovery, experimentalists continue to discover new surprising properties of FQH systems, and theorists continue to be fascinated by the rich structures revealed by FQH systems. Right now, we are still unsure if we understand every thing about FQH systems and unsure if there are new worlds waiting to be discovered. In my point of view, we are still far away from a complete understanding of FQH systems. We have only explored a small corner of a very rich and fascinating garden.

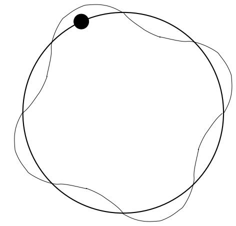

Figure 1: A particle and its quantum wave on a circle.

Partly due to our lack of complete and deep understanding of FQH systems, it has been difficult to explain to general audience, or even to physicists from a different field, what is so special about FQH effect, why FQH effect continues to sustain active research 18 years after its discovery. In this article I will offer one point of view on the importance of FQH effect. The FQH effect is important because
1) FQH states represent a whole new state of matter,
2) FQH states contain a whole new type of order,
3) FQH states expand and deepen our basic understanding of states of matter.

3 The global dancing pattern and the topological orders

Knowing that FQH liquids exist only at certain magical filling factors, such as 1/3, 2/5, 4/7, one cannot help to guess that FQH liquids should have some internal orders or "patterns". Different magical filling factors should be due to those different internal "patterns". The hypothesis of internal "patterns" can also help to explain the "rigidness" (*ie.* the incompressibility) of FQH liquids. Somehow a compression of a FQH liquid can break its internal "pattern" and cost a finite energy. However, the hypothesis of internal "patterns" appears to have one difficulty – FQH states are liquids, and how can liquids have any internal "patterns"?

Theoretical studies indeed reveal that it is possible to construct many different FQH states[4, 5, 6] presumably with different internal "patterns". It was realized that, however, these internal orders are different from any other known orders and cannot be observed in any conventional ways.[7] What is really new (and strange) about the orders in FQH liquids is that they are not associated with any symmetries (or the breaking of symmetries), and cannot be described by Landau's theory.[7, 8] This new kind of order is called *topological order*. Topological order is a new concept and we need a whole new theory to describe it.

To gain some intuitive understanding of topological order, let us first remind ourselves how do we describe a crystal order. Inside a crystal, atoms occupy fixed positions relative to other atoms. The crystal order is just a positional order that describes how atoms position themselves relative to other atoms. However, FQH states are liquids and the electrons do not have any positional order. But, the electrons do not just move randomly inside FQH states. They move around each other in a highly correlated manner. Such a correlated motion represents the internal structure of FQH liquids.

Let us try to visualize the correlated motion of electrons in a FQH state. A single electron in a magnetic field always move along circles (which are called cyclotron motions). Due to the wave property of the electron, the cyclotron motions are quantized such that the circular orbit must contain an integer number of the wave length. (See Fig. 1) We may regard the wave length as a step length and say that the electron always takes an integer number of steps to dance around the circle. If an electron takes n steps to go around the circle we say that the electron is in the n^{th} Landau level. At low temperatures, the electrons always stay in the first Landau level (and take one step around the circle), in order to have the lowest energy. When we have many electrons to form a 2DEG, electrons not only do their own cyclotron motion in the first Landau level, they also go around each other and exchange places (see a java simulation at *http://dao.mit.edu/~wen*). Those additional motions also subject to the quantization condition. For example an electron must take integer steps to go around another electron. The actual motions of electrons are a little more restricted due to the Fermi statistics of electrons:[9] an electron always take odd integer steps to go around another electron. (This fact actually explains why the filling factors almost always has an odd denominator.) Electrons in a FQH state not only move in a way that satisfies the quantization condition, they also try to stay away from each other as much as possible, due to the strong Coulomb repulsion and the Fermi statistics between electrons. This means that an electron tries to take more steps to go around another electron if possible.

Now we see that the quantum motions of electrons in a FQH state are highly organized. All the electrons dance collectively following strict dancing rules:
(a) all electrons do their own cyclotron motion in the first Landau level;
(b) an electron always takes odd integer steps to go around another electron;
(c) electrons always stay away from each other as much as possible.
We note that an electron does not just dance with one other electron, it dances with every other electron, since its motion is correlated with all other electrons. Thus the dancing pattern is a global dancing pattern in which all electrons dance together. We also see a fundamental difference between a correlated classical liquid and a correlated quantum liquid. In a correlated class liquid, the particle just try avoid each other [the condition (c)]. However, in a correlated quantum liquid, the particles also have to make sure to take an integer number of steps to go around each other. It is this quantum effect that makes FQH liquids very different from other more familiar classical liquids.

If every electrons follows these strict dancing rules, then only one unique global dancing pattern is allowed. Such a dancing pattern describes the internal quantum motion in the FQH state. It is this global dancing pattern that corresponds to the topological order in a FQH state. Different FQH states are distinguished by their different dancing patterns (or equivalently, by their different topological orders).

The simplest FQH state is a $\nu = 1/m$ Laughlin state in which an electron always takes exactly m steps to go around another electron. An compression of FQH liquids makes some electrons to take less than m steps around some other electrons. Such a break of dancing pattern costs finite energies and explains the incompressiblilty of FQH liquids. We see that the incompressiblilty is a quantum effect caused by the quantization of the dancing steps m. We cannot continuously reduce the distance between electrons by continuously reducing m, since m must be an odd integer.

A Laughlin state contains only one component of incompressible fluid. More general FQH states with filling factors such as $\nu = 2/5, 3/7, ...$ contain several components of incompressible fluid (those states are called abelian quantum Hall states). The dancing pattern (or the topological order) in an abelian quantum Hall state can also be described in a similar way by the dancing steps. The dancing pattern can be characterized by an integer symmetric matrix K and an integer charge vector Q.[6] An entry of Q, Q_i, is the charge (in unit of e) carried by the particles in the i^{th} component of the incompressible fluid. An entry of K, K_{ij}, is the number of steps taken by a particle in the i^{th} component to go around a particle in the j^{th} component. All physical properties associated with the topological orders can be determined in term of K and Q. For example the filling factor is simply given by $\nu = Q^T K^{-1} Q$. In the (K, Q) characterization of FQH states, the $\nu = 1/m$ Laughlin state is described by $K = m$ and $Q = 1$, while the $\nu = 2/5$ state, which has two components, is described by $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

It is instructive to compare FQH liquids with crystals. FQH liquids are similar to crystals in the sense that they both contain rich internal patterns (or internal orders). The main difference is that the patterns in the crystals are static related to the positions of atoms, while the patterns in FQH liquids are associated with the ways in which electrons "dance" around each other. However, many of the same questions for crystal orders can also be asked and should be addressed for topological orders.

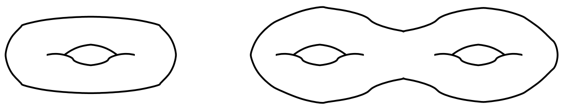

Figure 2: Riemann surfaces with genus $g = 1$ (a torus) and $g = 2$.

4 Measurement of topological orders

We know that crystal orders can be measured by X-ray diffraction. One important question is how do we measure the topological orders? Because the topological order is not associated with symmetries and not characterized by local order parameters, we need to find completely new ways to measure topological orders.

FQH liquids have a very special property. Their ground state degeneracy depends on the topology of space [10, 11, 7]. For example, the $\nu = \frac{1}{m}$ Laughlin state has m^g degenerate ground states on a Riemann surface of genus g [7]. For a more general abelian quantum Hall state characterized by integer matrix K, the ground state degeneracy is given by $(\det(K))^g$ on a Riemann surface of genus g. The ground state degeneracy in FQH liquids is *not* a consequence of symmetry of the Hamiltonian. It is robust against arbitrary perturbations (even impurities that break all the symmetries in the Hamiltonian)[7]. Only a change in the topological order (*ie* a switch in the dancing patterns) can induce a change in the ground state degeneracy. Thus the ground state degeneracy is a quantum number that can be used to characterize topological order and a measurement of the ground state degeneracy is a (partial) measurement of topological order.

We can understand the relation between the ground state degeneracy and the topological order through our dancing pattern description of the topological order. Somehow, the local dancing rules (a) – (c) uniquely determine the global dancing pattern only on a sphere. On a torus or other Riemann surfaces, there can be several global dancing patterns that satisfy the same local dancing rules. This results in several degenerate ground states. By determining how many global dancing patterns that can fit onto a Riemann surface, we can gain some information about the dancing pattern itself.

From our experience with crystal orders, we know the appearance of an order implies the existence of defects of the order. The kind of defects that can exist and the structure of the defects depend on the order in the parent state. Thus we can study an order by examining its defects. The same strategy applies to topological orders in FQH liquids. The defects of a topological order are called quasiparticles. They are excitations in the corresponding FQH state. Due to the non-trivial topological orders in FQH states, the quasiparticles have some most unusual properties. The quasiparticles can carry a fractional

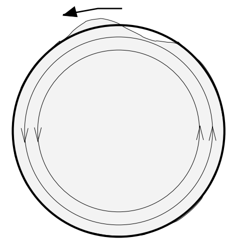

Figure 3: An edge wave on a FQH droplet propagates only in one direction.

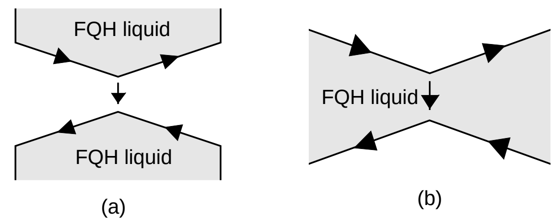

Figure 4: (a) Electron tunneling between edges of two HQ liquids. (b) Quasiparticle tunneling between two edges separated by a FQH liquid.

charge and have a fractional statistics. Those quantum numbers reflect the internal structure of a quasiparticle and the internal structure of the parent FQH liquid that support such a quasiparticle. Thus the topological orders can be measured by measuring the fractional charge and statistics of the quasiparticles.

A more complete and more practical measurement of topological order can be achieved through edge excitation of FQH liquids. FQH liquids as incompressible liquids have a finite energy gap for all their bulk excitations. However, FQH liquids of finite size always contain one-dimensional gapless edge excitations, which is another unique property of FQH fluids. The structures of edge excitations are extremely rich which reflect the rich bulk topological orders. Different bulk topological orders lead to different structures of edge excitations. Thus we can study and measure the bulk topological orders through edge excitations.[8]

To understand edge excitations, let us start with the simple $\nu = 1/m$ Laughlin state. Although the FQH liquid cannot be compressed, a finite FQH droplet can always change its shape without costing much energy. Thus the edge excitations are nothing but the surface waves propagating on the edge of the droplet (see Fig. 3). A more general FQH liquid contains several components of incompressible fluid and each component can deform independently. Thus an FQH liquid with k incompressible components will have k branches of edge excitations. Here we see how a property of bulk topological order (the number of incompressible components) is reflected in a property of edge excitations (the number of edge branches).

In addition to the number of edge branches, the dynamical properties of edge electrons are also depend on the bulk topological order in a sensitive way. For example the tunneling conductance between two edges (Fig. 4a) is proportional to a power of absolute temperature, T^{2g-2}. The exponent g is quantized and depends only on the topological order. Thus measuring g will reveal information about the bulk topological order.

To have an intuitive understanding of the exponent g, let us add an electron to the edge of a $\nu = 1/m$ Laughlin state. The other edge electrons have to take m steps to go around the added electron. Thus for larger m, the added electron causes larger disturbance and it is harder to add it to the edge at low temperatures. This will result in a larger exponent g. In fact $g = m$ for $\nu = 1/m$ Laughlin state.

Several experimental groups have successfully measured g through tunneling conductance.[12] These experiments open the door for experimental study of the rich internal and edge structures of FQH liquids.

When two edges are separated by a fractional FQH liquid (Fig. 4b), we can also have quasiparticle tunnelings. The quasiparticle tunneling has an interesting property that the tunneling resistance decreases as a power of temperature as the temperature is lowered. This remind us tunnelings between superconductors which have zero resistance at low temperatures. The analogy goes beyond the DC transport. The noise spectrum of the quasiparticle tunneling contains a singular peak at a "Josephson" frequency $f_J = e^*V/h$ associated with the fractional charge e^* of the quasiparticle. Such a singular peak in the noise spectrum is very similar to the AC Josephson effect in tunneling between superconductors. It would be very interesting to observe those fascinating properties in experiments, which will, at least, allow us to directly measure the fractional charge of the quasiparticles. If we can measure the noise power spectrum near the "Josephson" frequency $f_J = e^*V/h$, we can even determine the fractional statistics of the tunneling particles.[8]

5 What really is topological order

We have seen that FQH liquids are very different from other states of matter in the sense that they contain a new kind of order – topological order. The internal structures of FQH liquids cannot be characterized by symmetries, in contrast to other states of matter. Now the question is that why FQH liquids are so special. What is missed in Landau's theory for states of matter?

When we talk about orders in FQH liquids, we are really talking about the internal structure of FQH liquids at *zero* temperature. In other words, we are talking about the internal structure of the quantum ground state of FQH systems. So the topological order is a property of ground state wave function. The Landau's theory is developed for system at finite temperatures where quantum effects can be ignored. Thus one should not be surprised that the Landau's theory does not apply the states at zero temperature where quantum effects are important. The very existence of topological orders suggests that the finite-temperature orders and zero-temperature orders are different, and the zero-temperature orders contain richer structures. Now it is clear that what is missed by Landau's theory is the quantum effect.

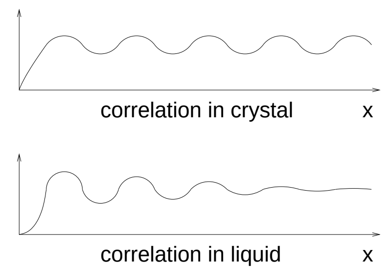

Figure 5: Correlations in crystal, $G_c(x)$, and in liquid, $G_l(x)$.

To gain a deeper understanding of topological order, let us examine more carefully the orders in ordinary states of matter (*ie* at finite temperatures where quantum effects can be ignored). We will call those orders classical orders to distinguish them from the topological order. At finite temperatures, the full description of a system is given, mathematically, by a probability distribution. To describe the positional order of particles in a system, we can use the probability distribution $P(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_N)$ where $\boldsymbol{r}_i$ is the coordinate of the i^{th} particle and N is the total number of the particles. As we change temperature, pressure, and other external conditions, the probability distribution P changes continuously. However, the systems described by those different distributions can have very similar properties and describe the same phase. We group all those similar probability distributions into a single class, which is called a universality class. If we change the external conditions too much, it can cause an abrupt change in the properties of the system. In this case we say there is a phase transition to another phase which is described by a different classical order. The probability distributions in different phases belong to different universality classes. As a concrete example of the above general discussion, let us consider the phase transition between a liquid and a crystal, described by the distributions P_l and P_c respectively. To understand the qualitative difference between the two distributions, we introduce the correlations

$$\begin{aligned} G_l(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \int \prod_{i=3}^{N} d\boldsymbol{r}_i \; P_l(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, ..., \boldsymbol{r}_N) \\ G_c(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \int \prod_{i=3}^{N} d\boldsymbol{r}_i \; P_c(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, ..., \boldsymbol{r}_N) \end{aligned}$$

which are shown in Fig. 5. We see that the oscillations in

G_c continue indefinitely, indicating a long range crystal order.

Through the above discussions, the following points become clear:
(i) A classical order is a property of the probability distribution $P(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_N)$ in the $N \to \infty$ limit. A classical order describes the structures in *positive* functions with infinite number of variables.
(ii) Similar distributions (*i.e.* the distributions in the same universality class) have the same classical order. The classical order is really a property of universality classes, rather than a property of individual distributions.
(iii) It is a deep insight to realize that the different universality classes are determined by the symmetry properties of the distributions. Therefore the classical orders (and the universality classes) are characterized by the symmetries. This is the foundation of Landau's theory of (classical) orders and phase transitions (at finite temperatures).

Strictly speaking, althrough the Landau's theory and the symmetry description of orders and phase transitions represent an important mile stone in our understanding of orders, the theory actually cannot describe all the classical orders. This is because some classical phase transitions, such as the Kosterliz-Thouless transition, do not change any symmetries. Thus despite the success of the Landau's theory, even some classical orders are not fully understood.

Now let us examine carefully the orders in zero temperature states. We will call those orders quantum orders. First, the quantum orders are properties the ground state wave functions of the system. We immediately see that the quantum orders and the classical orders have parallel mathematical structures. A classical order is a property of probability distribution P which is a *positive* function of N coordinates of the particles. A quantum order is a property of ground state wave function ψ which is a *complex* function of N coordinates of the particles. The distribution P and the wave function ψ is related

$$P(\boldsymbol{r}_1, ..., \boldsymbol{r}_N) = |\psi|^2(\boldsymbol{r}_1, ..., \boldsymbol{r}_N)$$

Through this relation we see that we can use classical order (which is characterized by symmetries) to describe, at least partially, the internal structure of ground state wave function. Therefore, we can use symmetries to partially characterize the internal structure of ground state wave function. It is also clear that the characterization by classical order misses the phase of the wave function which is a quantum effect. Thus it is possible that certain structures in ground state wave functions cannot be described by classical orders. Quantum orders provide a full description of those structures and in general contain richer structures than classical orders.

Just like classical orders, quantum orders characterize universality classes of ground state wave functions. If we change the interaction between the particles, the ground state wave function changes continuously. If those ground states have similar properties, we say they describe the same phase and belong to the same universality class. Those states have the same quantum order. However, changing the interaction by a large amount may cause abrupt changes in ground state properties. In this case we say the change of interaction drives the ground state wave function from one universality class to another, which leads a phase transition and a change of quantum order.

In a sense, the classical world described by positive probabilities is a world with only "black and white". The Landau's theory and the symmetry principle for classical orders can describe different "shades of grey" in the classical world. The quantum world described by complex wave functions is a "colorful" world. The Landau's theory and the symmetry principle cannot describe the "colors". We need to use new theories, such as the theory of quantum order, to describe the rich "color" of quantum world. It is clear that the quantum orders are much richer then the classical orders.

The topological order introduced before is a property of two dimensional electron gases at zero temperature. Thus the topological order is special kind of quantum order. In simple cases, a topological order is a quantum order where all the excitations above ground state have finite energy gapes.

Now it is clear why we need topological orders to characterize FQH liquids. Different FQH liquids have the same symmetries. Thus we cannot use symmetries and local order parameters to distinguish different FQH liquids. However, when we examine the quantum orders in FQH liquids (*ie* when we examine the ground state wave function ψ instead of the absolute value square of the wave function $|\psi|^2$), we find different FQH liquids carry different quantum orders. This allows us to understand in which sense different FQH liquids are different.

It is also clear that the topological orders (and the quantum orders) are general phenomena. They do not just appear in some special states, such as FQH states. Topological orders and quantum orders are general properties of any states at zero temperature. Non trivial topological orders not only appear in FQH liquids, they also appear in spin liquids at zero temperature. In fact, the concept of topological order was first introduced in a study of spin liquids.[7] FQH liquid is not even the first experimentally observed state with non trivial topological orders. That honor goes to superconducting state discovered in 1911.[13] Superconducting states contain non trivial topological orders[14]. In contrast to common point of view, a superconducting state is fundamentally different from a superfluid state and cannot be characterized by breaking symmetries.

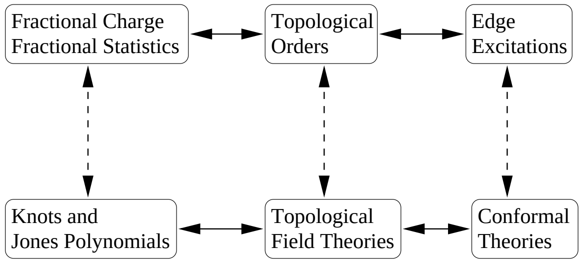

Figure 6: Relation of FQH theory with string theory and some modern mathematics.

6 Mathematical structures behind the topological orders

The topological orders in FQH states are extremely rich (much richer than crystal orders), and can be measured experimentally by edge excitations. Despite our complete understanding of the topological orders and edge excitations in abelian quantum Hall states, we are still lack of a theory for topological orders in most general FQH states. We do know some examples of FQH states (called non-abelian quantum Hall states[15]) that are beyond our "dancing-step" description and cannot be characterized by the K-matrix and the charge vector. We have a general theory of crystal orders because we know the mathematical frame work – the group theory – behind the crystal orders. However, after 18 years of theoretical research, we still do not know the mathematical frame work behind the topological orders (or more general quantum orders), and we still do not have a general theory for topological orders in FQH states.

The theoretical studies so far do reveal some fascinating mathematical structures in FQH liquids,[15] which allow us to peek into some aspects of the final mathematical frame work behind the topological orders.[16] It is quite amazing that the mathematical structures found in FQH liquids are directly related to some modern mathematics developed in last century, and some of them are even under developments right now. The similar mathematical structures also appeared and developed in string theory. It is so rare in modern physics to have an experimental phenomenon that tie so closely to some modern mathematics. Thus FQH theory draw interests of researchers from very different fields, and, in my opinion, will have impact on those fields, in particular on some branches of modern mathematics.

Fig. 6 illustrate some close ties between FQH theory, modern mathematics and the mathematics developed in the string theory. Witten developed topological theory in his study of string theory,[17] and used the topological theory to study the Jones polynomial in a mathematical theory of knots. The study generated such an impact in mathematics that Witten, as a physicist, won the Fields metal in 1990. Now we understand that the topological theories are closely related to the topological orders. In fact, the topological theories are the low energy effective theories of topological orders.[7, 8] The FQH liquids are the experimental realizations of topological theories. The close relation between topological theories and knots exactly mirror the close relation between topological orders and fractional statistics. Witten also pointed out a close relation between topological theories and conformal theory, which exactly mirror the close relation between topological orders and edge excitations. These connections are not accidental. They are signs of a more general and more coherent picture behind FQH liquids.

7 No end to the richness of the nature

When looking back, it is hard to believe that the discovery by Tsui, Stormer, and Gossard 18 years ago has such a deep impact, and its potential has not exhausted yet. FQH effect opens up a whole new territory for physical explorations, both in experiments and in theory. We learned so much about FQH systems in last 18 years, and yet those progresses lead to deeper questions that require further studies. The studies of topological orders in FQH liquids will deepen our understanding on orders, and may lead to discoveries of new states of matter beyond FQH systems.

There is no end to the richness of the nature.

8 Acknowledgments

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