TOPOLOGICAL ORDERS AND CHERN-SIMONS THEORY IN STRONGLY CORRELATED QUANTUM LIQUID *

XIAO-GANG WEN

Institute for Advanced Study
Princeton, NJ 08540

ABSTRACT

We review the topological orders in strongly correlated quantum liquids. The characterization of the topological orders through ground state degeneracy, non-Abelian Berry's phases and edge excitations are discussed.

It becomes more and more clear that some strongly correlated quantum liquids, such as the quantum Hall states and the chiral spin liquid states, contain extremely rich internal structures (topological orders).[1,2,3,4,5,6] Such rich internal structures are reflected in the complexity of the quasiparticle quantum numbers and the edge state structures. It was shown that the FQH states and the chiral spin states be classified by symmetries and the associated order parameters. Therefore they represents new kinds of universality classes not related to broken symmetries.[3,4] In order to describe those strongly correlated quantum liquids the concept of "ordering" need to be generalized.

In this article I will briefly review recent progresses in understanding the internal structures of the FQH states, chiral spin states and anyon superfluid states. I will also discuss the relation between the topological Chern-Simons theories and the topological orders in the FQH states and the chiral spin states.

1. Ground state degeneracy and the topological orders

Before going into a general discussion of the concept of topological orders, let us first study some novel properties of FQH states. As an example we will concentrate on the simple Laughlin state with filling fraction $\nu = \frac{1}{q}$.

It is well known that the Laughlin states are non-degenerate on sphere,[1] but have q fold degeneracy if was put on torus.[7,4] There are two simple ways to understand the degeneracy of the FQH states on the torus.

Consider a FQH state on a torus. We adiabatically add an unit flux through the hole of the torus. The Hamiltonian is unchanged after adding an unit flux. Therefore the adiabatic process induces a transformation between the ground states of the same

* *Int. J. Mod. Phys.* **B5**, 1641 (1991)

Hamiltonian. Such a transformation can be represented by an unitary operator U_1 which acts on the ground states. Similarly, the adiabatic turning on an unit flux going through the tube of the torus generates an unitary transformation U_2 acting on the ground states. It was shown that the commutator of the operators U_1 and U_2 are determined by the Hall conductance[8]

$$U_1 U_2 = e^{i2\pi \frac{\sigma_{xy} h}{e^2}} U_2 U_1 = e^{i\frac{2\pi}{q}} U_2 U_1 \tag{1}$$

The ground states must form a representation of the above algebra (Heisenberg algebra). The Heisenberg algebra has only one irreducible representation which is q dimensional. Therefore the ground state degeneracy (GSD) of the $\nu = 1/q$ Laughlin state must be a multiple of q.

In the second proof, the GSD is directly related to the fractional statistics.[4,9] We know the Laughlin state supports quasiparticles with fractional statistics $\theta = \frac{\pi}{q}$. To relate the fractional statistics to the GSD, let us consider the following tunneling process. A pair of quasiparticle and anti-quasiparticle is created at a certain time. The quasiparticle propagates in $\hat{x}$ direction all the way around the torus and then annihilates with the anti-quasiparticle. Such a tunneling process induces a transition between ground states. The transition can be represented by an unitary operator T_1: $|\Psi_0'\rangle = T_1|\Psi_0\rangle$. If we let the anti-quasiparticle to go all the way around the torus in the $\hat{x}$ direction we will obtain a different transition operator $\bar{T}_1$. However, $\bar{T}_1$ must be equal to T_1^{-1} because the two tunneling processes cancel each other. Similarly, we can obtain another transition operator T_2 by letting the quasiparticle propagates in the $\hat{y}$ direction.

Now let us consider a sequence of four tunneling processes described by T_1, T_2, T_1^{-1} and T_2^{-1}. Notice that the four tunneling paths can be deformed into two linked loops which give rise to a pure phase $e^{-i2\theta}$ as implied by the fractional statistics of the quasiparticles. Therefore we have

$$T_2^{-1} T_1^{-1} T_2 T_1 = e^{-i2\theta} = e^{-i\frac{2\pi}{q}}. \tag{2}$$

Which is again the Heisenberg algebra. Since the ground states form a representation of (2), we find that the ground states must be (at least) q-fold degenerate.

The above argument can be generalized to high genus Riemann surfaces.[4] A genus g Riemann surface can be regarded as g torus connected by tubes. Each torus will generate a Heisenberg algebra (2). The algebra generated by different torus commute with each other. Therefore the Laughlin state have q^g fold degeneracy on a genus q Riemann surface.

We would like to point out that the above two proofs are very general. They are valid for any local Hamiltonians as long as the gap remain finite. Therefore GSD remains even in presence of random potentials or any other static perturbations. The existence of the GSD and robustness of the GSD really reflect the new kind of ordering in the FQH states.

The GSD in the FQH states can not be a consequence of broken symmetries[4] (such as the formation of CDW). This is because of the following reasons. A) the GSD depend on the topology of the space. B) The GSD is robust against *any* perturbations. If the degeneracy was due to broken symmetry, the degeneracy could be lifted by a symmetry breaking perturbation. C) For a finite FQH system the energy split between different degenerate ground states is of order $e^{-L/\xi}$ where L is the linear size of the system and ξ is the typical length scale of the FQH state. This result is derived from the fact that the different ground states of the FQH system can be connected by the quasiparticle tunneling process discussed above. If the GSD was due to broken discrete symmetry, the energy split between different ground states could be at most of order e^{-L^2/ξ^2}. This is because the different ground states of broken symmetry phase can only be connected through domain wall tunneling.

The robustness of the GSD reflects some sort of ordering in the FQH states. The dependence of the GSD on the topology of the space further indicates that the ordering has a long range nature. Because the degeneracy of the FQH ground states is not due to symmetry breaking, we are forced to introduce a new concept – topological order – to describe the special ordering in the FQH states. The GSD discussed above can be regarded as a defining property of the topological orders. Or in the other words, the GSD can be regarded as a quantum number that characterize different topological orders.

The concept of the topological order is not limited in FQH states, and is not limited in two spatial dimensions. Let us consider a rigid state in d (spatial) dimensions. By rigid state we mean a state contains on gapless quasiparticle excitations. At low energies (below the gap), the only non-trivial feature in the rigid state is the GSD. From the above example we see that the GSD does not always come from broken symmetries. To further characterize the properties of the degenerate ground states, let us consider how the ground states tunnel into each other in a finite system. If the degenerate ground states come from broken symmetries, the ground states can tunnel into each other through the domain wall tunneling. The domain wall is a defect in the ordered state and has a dimension $d-1$. The energy split between ground states is of order e^{-L^d/ξ^d}. However it is possible that the ground states can tunnel into each other through creation of a $d_t - 1$ dimensional defect. In this case the energy split of the ground states will be of order $e^{-L^{d_t}/\xi^{d_t}}$. If $d_t \neq d$, the rigid state will represents a new kind of universality classes which can not be classified by symmetries. In this case we can say that the rigid state contains a non-trivial topological order. We will call d_t the dimension index (DI) of the topological order. The DI of the topological orders in the FQH states is equal to 1. The short range RVB state discussed in Ref. 10 is an example of topologically ordered state in arbitrary spatial dimensions with DI=$d-1$.

According to our definition the topological orders do not exist in one dimension. We also see that DI=1 topological orders do not exist at finite temperatures (due to the same reason that the two dimensional superconducting state do not exist at finite temperatures). The topological order with DI≥ 2 are expected to exist at finite temperatures. We also expect a phase transition between topologically ordered state and the high temperature disordered state.

In three dimensions DI may be equal to 1 or 2. The DI=2 topological order is realized in the short range RVB state and the BCS superconducting state (when the gauge field is treated dynamically). We do not know any three dimensional system which has a DI=1 topological order. It would be interesting to find such systems.

2. Topological orders in the FQH states and Chern-Simons theory

We know an universality class describes a class of systems which flow to the same infrared fixed point. For every infrared fixed point we have a so called low energy effective theory to describe the systems at or near the fixed point. The characteristic effective theory of the topologically ordered FQH states is nothing but the topological Chern-Simons theory.

The low energy effective theory at the infrared fixed point is much simpler than the original high energy theory. The effective theory of the topologically ordered state, containing only finite degrees of freedom at low energies (a consequence of the gap), is the simplest field theory. The field theory with finite number of degrees of freedom is called topological theory and has attracted a lot of attention after Witten's pioneer work.[11]

It turns out that generic FQH states can be divided into Abelian FQH (A-FQH) states and Non-Abelian FQH (NA-FQH) states. The A-FQH states only contain quasiparticles with Abelian fractional statistics, while the NA-FQH states may contain quasiparticles with non-Abelian statistics. It was shown[5,6] that the A-FQH states are classified by integer valued symmetric matrices K (up to an equivalence condition $K \sim O^T K O$ where $O \in SL(N, Z)$ is a integer valued matrices of unit determinant). The effective Chern-Simons theory of the A-FQH states has a form

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} \epsilon^{\mu\nu\lambda} a_{I\nu} \partial_\nu a_{J\lambda}. \tag{3}$$

where a_I are fictitious $U(1)$ gauge fields. The quasiparticles carry unit charge of the fictitious gauge fields. All the quasiparticle quantum numbers, such as statistics and charges, can be obtain from the matrices K.[5] The hierarchy states constructed by Haldane and Halperin can be shown to be characterized tri-diagonal matrices. The FQH states constructed by Jain in general are characterized by more complicated matrices. However using the above equivalence relation one can show that some of the Jain's states are equivalent to the Haldane-Halperin hierarchy states (as long as the topological orders are concerned).

Now let us address a physical question regarding the effective Chern-Simons theory of the FQH states. We know the effective theory (3) is not some thing we can directly measure. Thus it is not guaranteed that all the features in the effective theory are physical and measurable in experiments. Some times the effective theory may contain features that are not observable. In this case two different effective theories may describe the same physical state.[12] With these possibilities in mind we would like to ask "is the matrix K physical? Can we measure K in experiments?"

First the determinant of K is physical and easy to measure because $\det K$ is equal to the GSD. To physically measure other entries of K is less trivial. One way to measure K is to use the non-Abelian Berry's phases.[3]

Let us consider an electron system in magnetic field:

$$H_\tau = -\frac{1}{2}m_{ij}^{-1}(\tau)\left(\frac{\partial}{\partial x_i} - ieA_i\right)\left(\frac{\partial}{\partial x_j} - ieA_j\right) \tag{4}$$

where $m_{ij}(\tau)$ is the mass matrix of the electrons:

$$\left(m_{ij}\right)^{-1} = \frac{1}{m_0}\begin{pmatrix} 1+\left(\frac{\text{Re }\tau}{\text{Im }\tau}\right)^2, & -\frac{\text{Re }\tau}{(\text{Im }\tau)^2} \\ -\frac{\text{Re }\tau}{(\text{Im }\tau)^2}, & \frac{1}{(\text{Im }\tau)^2} \end{pmatrix} \tag{5}$$

and τ is a complex number. (4) describes a family of the electron systems labeled by τ. Let us assume that the electrons in H_τ form a FQH state described by (3). Then for each τ, H_τ has $|K|$ fold degenerate ground states $|\Phi_n(\tau)\rangle$, $n = 1, ..., |K|$ where $|K| = \det K$. We notice that H_τ and $H_{\tau+1}$ actually describe the same system, because $m^{-1}(\tau)$ and $m^{-1}(\tau+1)$ are related by a coordinate transformation $x_1 \to x_1 - x_2,\ x_2 \to x_2$. Similarly one can show that H_τ and $H_{-1/\tau}$ describe the same system due to the transformation $x_1 \to x_2,\ x_2 \to -x_1$. Therefore $|\Phi_n(\tau)\rangle$, $|\Phi_n(\tau+1)\rangle$ and $|\Phi_n(-1/\tau)\rangle$ span the same Hilbert space and we can choose that

$$|\Phi_n(\tau)\rangle = |\Phi_n(\tau+1)\rangle = |\Phi_n(-1/\tau)\rangle \tag{6}$$

The non-Abelian Berry's phase is an unitary matrix that is associated with an adiabatic deformation of the Hamiltonian H_τ.[13] The deformation starts and ends with the same Hamiltonian. Let us denote the deformation path by $\tau(t)|_{t=0}^{1}$. Then the matrix of the non-Abelian Berry's phase is given by

$$W[\tau(t)] = P\exp[-i\int_0^1 A(t)dt] \tag{7}$$

where P denotes the path ordered product and A is a matrix defined by

$$A_{nm}(t) = i\langle\Phi_n[\tau(t)]|\ \frac{d}{dt}\ |\Phi_m[\tau(t)]\rangle \tag{8}$$

The non-Abelian Berry's phases induced by the FQH states have the following special properties.[11,3] For the path starts and ends with the same τ (*i.e.*, $\tau(0) = \tau(1)$) the non-Abelian Berry's phase (denoted as $W(\tau,\tau)$) is a pure phase:

$$W(\tau,\tau)_{mn} = e^{i\theta}\delta_{mn} \tag{9}$$

The value of θ may differ from path to path. If the path connects τ and $\tau+1$ (*i.e.*, $\tau(1) = \tau(0)+1$) then the corresponding non-Abelian Berry's phase (denoted as $W(\tau,\tau+1)$) is longer a pure phase. However only the phase of $W(\tau,\tau+1)$ depends on different choices of paths connecting τ and $\tau+1$. Thus $W(\tau,\tau+1)$ can be written in the following form

$$W(\tau,\tau+1) = e^{i\theta}U \tag{10}$$

where $U \in SU(|K|)/Z_{|K|}$ is independent of the paths. Similarly the non-Abelian Berry's phase $W(\tau,-1/\tau)$ associated with paths connecting τ and $-1/\tau$ has a form

$$W(\tau,-1/\tau) = e^{i\theta}S \tag{11}$$

Again $S \in SU(|K|)/Z_{|K|}$ is independent of the paths.

Because the two matrices U and S are path independent, they reflect the intrinsic properties of the FQH state. U and S, as $|K|\times|K|$ matrices, contain a lot of information about the topological orders. In particular they contain the information about the matrix K. Now we can say that the matrix K can be measured through the non-Abelian Berry's phases U and S.

We would like to mention that the two transformations $\tau \to \tau+1$ and $\tau \to -\frac{1}{\tau}$ generate so called moduli group

$$\tau \to \frac{a\tau+b}{c\tau+d} \tag{12}$$

where $a,b,c,d,\in Z$ are integers and $ad-bc=+1$, *i.e.*,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z). \tag{13}$$

The pair $\{U,S\}$, being associated with the generators of $SL(2,Z)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, generates a $|K|$ dimensional projective representation of the moduli group $SL(2,Z)$. We see that the topological orders in the FQH states are closely related to the projective representations of the moduli group.

We would like to point out that the eigenvalues of U have a form $e^{i2\theta_n}$ which determine the allowed quasiparticle statistics θ_n in the A-FQH state. Thus quasiparticle statistics can be measured through the non-Abelian Berry's phases without even creating the quasiparticles. This relation may be useful in numerical calculations.

The whole discussion in the above can be generalized to high genus Riemann surfaces. Again the non-Abelian Berry's phases are given by the representations of the moduli group for high genus Riemann surfaces. The non-Abelian Berry's phases for high genus Riemann surfaces may contain even more informations about the topological orders. We would like to conject that the non-Abelian Berry's phases on all the Riemann surfaces completely characterize the topological orders (hence the FQH states) in 1+2 dimensions.[14]

Recently it was realized that the topological orders described by the non-Abelian Chern-Simons theories also exist in FQH systems.[15] A class of NA-FQH states is described by the following wave function

$$\chi_1^k(z_i)[\chi_M(z_i)]^N \tag{13}$$

which has a filling fraction $\nu = \frac{M}{kM+N}$. χ_m in (13) is the wave function of m filled Landau levels. The charge $\frac{e}{kM+N}$ quasiparticles in such a NA-FQH state can be shown to have a non-Abelian statistics described by the level M $SU(N)$ Chern-Simons theory. What is more interesting is that the wave function (13) can be shown to be exact ground state of some *local* Hamiltonians.

From the above discussion we see that the topological orders in the FQH states have extremely rich structures. It is very important to find a way to classify all the possible topological orders in the FQH states. Due to the similarity between the chiral spin states, the anyon superfluid states and the FQH states, the above discussion also apply to the former two cases.[12]

3. Edge states – a practical way to measure the topological orders

The GSD and the non-Abelian Berry's phases discussed in last section can only be used in numerical calculations to detect the topological orders. In real experiments the FQH samples always have boundaries and do not form a closed Riemann surface. In this case we can use the edge excitations[16,17,18,19] to measure the topological orders in the bulk. The edge excitations in the FQH states are also found to have extremely rich structures. For the A-FQH states described by (3), the edge excitations are described by the $[U(1)]^{|K|}$ Kac-Moody algebra.[16,20,5] The edge excitations of the NA-FQH states in (13) are described by $U(1) \times SU(M)$ Kac-Moody algebra of level N.[15] The electron operators on the edge correspond to primary fields of the Kac-Moody algebra.[21,19,20,15] The topological orders in the bulk states and the structures of the edge excitations have an one to one correspondence. Thus the bulk topological orders can also be classified through the edge excitations. The correspondence between the topological orders and edge excitations is just the physical realization of the correspondence between the Chern-Simons theory and conformal theory discovered by Witten.[11] However there is one difference. The edge excitations in the FQH states do not respect conformal symmetry. They even do not respect Lorentz symmetry because in general the edge excitations contain several different velocities.

Recent advance in fabrication of small QH device make it feasible to study in detail the properties of the edge excitations in the FQH states. The experimental explorations of the topological orders in the FQH states may became quite realistic in near future. Thus it is very important to have a good understanding about the dynamical properties of the edge excitations. In particular it is very interesting to find some experimental predictions that distinguish the NA-FQH states from the A-FQH states and to try to search for possible NA-FQH states in real samples. The NA-FQH states are likely to exist in the samples in which spin reversed state or several different Landau levels play an important role in formation of the FQH states.

4. Acknowledgment

This research is supported by DOE grant DE-FG02-90ER40542.

5. References

1. F.D.M. Haldane,Phys. Rev. Lett. 51,605(1983).
2. B. I. Halperin, Phys. Rev. Lett.,52, 1583 (1984).
3. X.G. Wen, *Int. J. Mod. Phys. B* **2**, 239 (1990); *Phys. Rev.* **B40**, 7387 (1989).
4. X.G. Wen and Q. Niu, *Phys. Rev.* **B41**, 9377 (1990).
5. B. Blok, and X.G. Wen, *Phys. Rev.* **B42**, 8145 (1990); *Phys. Rev.* **B42**, 8133 (1990); Phys. Rev. B to appear.
6. N. Read, Phys. Rev. Lett., 65 (1990) 1502
7. F.D.M. Haldane and D. Rezayi, *Phys. Rev.* **B31**, 2529 (1985); F.D.M. Haldane, *Phys. Rev. Lett.* **55**, 2095 (1985).
8. Q. Niu, D.J. Thouless, *J. Phys.* **A17**, 2453 (1984); Q. Niu, D.J. Thouless and Y.S. Wu, *Phys. Rev.* **B31**, 3372 (1985).
9. X.G. Wen, E. Dagotto and E. Fradkin, *Phys. Rev.* **B42**, 6110 (1990).
10. X.G. Wen, IAS preprint, to appear in Phys. Rev. B; see also N. Read and B. Chakraborty, *Phys. Rev.* **B40**, 7133 (1989).
11. E. Witten, *Comm. Math. Phys.* **121**, 351 (1989).
12. X.G. Wen and A. Zee, "Topological orders, Universality Classes and Statistical Screening in the Anyon Superfluids" ITP preprint.
13. F. Wilczek and A. Zee, *Phys. Rev. Lett.* **52**, 2111 (1984).
14. A more precise way to state the conjecture can be found in Ref. 3.
15. X.G. Wen, *Phys. Rev. Lett.* **66**, 802 (1991).
16. X.G. Wen, NSF-ITP-89-157, ITP preprint (1989), to appear in Phys. Rev. B.; *Phys. Rev. Lett.* **64**, 2206 (1990); D.H. Lee and X.G. Wen, IBM and IAS preprint.
17. A.H. MacDonald, *Phys. Rev. Lett.* **64**, 220 (1990).
18. F.D.M. Haldane, *Bulletin of APS* **35**, 254 (1990).

19. M. Stone, "Edge waves in the QH effect", IL-TH-8, Urbana preprint (1990); "Vertex Operators in the QH effects", IL-TH-90-32, Urbana preprint (1990). A. Balatsky and M. Stone, Urbana preprint (1990).

20. X.G. Wen, *Mod. Phys. Lett. B* **5**, 39 (1991).

21. X.G. Wen, *Phys. Rev.* **B41**, 12838 (1990).