Ground state degeneracy of the FQH states in presence of random potential and on high genus Riemann surfaces†

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ABSTRACT: The FQH states are shown to have $\tilde{q}^g$ fold ground state degeneracy on a Riemann surface of genus g, where $\tilde{q}$ is the ground state degeneracy in a torus topology. The ground state degeneracies are directly related to the statistics of the quasi-particles given by $\theta = \frac{\tilde{p}\pi}{\tilde{q}}$. The ground state degeneracy is shown to be invariant against weak but otherwise arbitrary perturbations. Therefore the ground state degeneracy provides a new quantum number, in addition to the Hall conductance, characterizing different phases of the FQH systems. The phases with different ground state degeneracies are considered to have different topological orders. For a finite system of size L, the ground state degeneracy is lifted. The energy splitting is shown to be at most of order $e^{-\frac{L}{\xi}}$. We also show that the G-L theory of the FQH states (in the low energy limit) is a dual theory of the $U(1)$ Chern-Simons topological theory.

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I. INTRODUCTION

There are two quantum fluid states which are known to exist at zero temperature, *i.e.,* they may appear as the ground state of a system. One is superfluid and the other is incompressible fluid. Superfluid state was first discovered in He^4 (1932)[1] and later in He^3 (1972).[2] The first example of incompressible fluid state is probably the superconducting state[3] discovered in 1911. The superconducting state is incompressible if we fix the positive background charge density which comes from lattice ions (by assuming the lattice to be rigid). All excitations in the superconducting state have finite energy gaps (except phonons which have been excluded). The incompressibility of the superconducting state comes from the long range Coulomb interaction. In the early 1980's a new class of incompressible quantum fluid was discovered in the Integer Quantum Hall (IQH) effects and in the Fractional Quantum Hall (FQH) effects.[4] Recently in studying high T_c superconductors, a class of "incompressible" quantum spin liquid states – chiral spin states – was proposed,[5,6] which support no gapless excitations. The time reversal symmetry (T) and the parity (P) are broken in these spin liquid states. Chiral spin states are closely related to the FQH states.[6,5]

The FQH states and chiral spin states are very special in the sense that their ground state properties are not characterized the symmetries in the ground states. The transition from one FQH state (or chiral spin state) to another is not associated with a change in the symmetries of the states. In this paper we will demonstrate that the FQH states and chiral spin states contain non-trivial topological structures. The different FQH states and chiral spin states may be classified by topological orders.

It has been shown that the topological orders in chiral spin states can be partially characterized by the ground degeneracy of chiral spin states in compactified space.[7,8] The ground state degeneracy depends on the topology of the space and is equal to k^g (ignoring the two fold degeneracy arising from the spontaneous T and P breaking), where g is the genus of the compactified space and k is an integer characterizing the topological order in chiral spin states. Similarly, it is known long ago that the ground state degeneracy of the FQH states also depend on the topology of compactified space. For the simplest FQH states given by Laughlin wave function

$$\psi(z_i) = \left[\prod_{i<j}(z_i - z_j)^q\right] e^{-\frac{1}{4}\sum_i |z_i|^2} \tag{1.1}$$

the ground state is found to be non-degenerate on a sphere[9] and q fold degenerate on a torus[10] (with $g = 1$). The dependence of the ground state degeneracy on the topology of space suggests that the FQH states also contain non-trivial topological orders.

The ground state degeneracy of the FQH states has been a puzzling problem for a long time. Especially, it is not clear whether the degeneracy arises from broken symmetries or not. There are arguments both favoring and disfavoring the symmetry breaking picture.

According to Anderson,[11] some basic ingredients of symmetry breaking are already contained in the Laughlin's description of the FQH system on a circular disc. In the 1/3-filling case, for instance, the Laughlin states with zero, one and two quasi-holes are macroscopically distinct, but energetically they are the same for the bulk of the system. The Laughlin state with three quasi-holes differs from that of no quasi-hole by only one single particle state and they are therefore macroscopically indistinct.

Tao and Wu[12] considered the case of a cylinder geometry and concluded, using general gauge symmetry arguments similar to those of Laughlin for the FQH case, that there must be a symmetry breaking of the system in order to exhibit the FQH effect. The same conclusion was reached by Niu, *et al.*,[13] who studied the problem from a point of view of the topological invariant of the quantum Hall conductance on a torus. The latter authors also demonstrated the degeneracy of the ground states by explicit construction of distinct Laughlin states on the torus. The existance of degeneracy was further supported by the numerical result of Su.[14] The generality of the arguemtns of gauge symmetry and topological invariance make the degeneracy a very robust property of the FQH system, independent of perturbations which do not close the energy gap.

There are also arguments disfavoring the idea of symmetry breaking. These are backed by the evidence of no ground state degeneracy in the sphere geometry.[9] The kind of degeneracy found in the torus geometry was interpreted as the degeneracy of center of mass motion,[10] and therefore does not qualify as degeneracy among macroscopically distinct ground states, which is essential for symmetry breaking.

In this paper we wish to resolve the above puzzle. We argue that the ground state degeneracy of the FQH states is really a reflection of the topological order of the system. The degeneracy depends on the topology of the system geometry, and is preserved (in the thermodynamic limit) even when the translational and rotational symmetries of the system are absent. Therefore, the degeneracy should not be interpreted as a symmetry breaking of the usual type, nor should it be regarded as the center of mass degeneracy.

If one insists on the symmetry breaking picture, one may attribute the ground degeneracy to broken "topological" symmetries (see Section 9). However, the topological symmetry can be defined only after the topology of space is specified. The very existence of the topological symmetries depend on the topology of the space. The number of the topological symmetries are different for the spaces with different topologies.

The characterization of the FQH states is another unresolved problem in the FQH theory. The Hall conductance is certainly not enough to characterize the different FQH states. Two different FQH states may give rise to the same Hall conductance and yet be macroscopically distinct. Because the ground state degeneracy of the FQH states is robust against arbitrary perturbations, the ground state degeneracy can be used to characterize different phases in phase space. Therefore the different FQH states with the same Hall conductance can be (at least partially) characterized (or distinguished) by their different ground state degeneracies (on torus and high genus Riemann surfaces). A more complete characterization of the topological orders in the FQH states can be obtained by studying the non-Abelian Berry's phases[15] associated with twisting the mass matrix of the electrons.[8]

The paper is arranged as follows. In Section 2 we study the ground state degeneracy on a torus and its lifting by impurity potentials using the first order perturbation theory. The selection rule of the magnetic translation group implies an energy splitting exponentially small in the shortest linear size of the system. In Section 3 and 4 we study the ground state degeneracy using the effective theory of the FQH states. The effective theory approach not only applies to case with spatial dependent magnetic field, random potentials, *etc.* , it also applies to high genus Riemann surfaces where the magnetic translations can not be defined. In Section 5 we study the splitting of the ground state energies of finite system based on the effective theory approach. Comparing with the results obtained in Section 2, the results obtained here are non-perturbative (but qualitative). The energy split is found to be of order $e^{-L\sqrt{m^*\Delta}}$ for generic random potentials. Here m^* and Δ are the effective mass and the energy gap of the fractionally charged quasi-particles and quasi-holes, and L

is the size of the system. We also demonstrate explicitly that the ground state degeneracy of the FQH state (or any other systems) is determined directly by the fractional statistics of the quasi-particles, instead of by the filling fraction $\nu = \frac{p}{q}$. In Section 6 the ground state degeneracy of the hierarchy FQH states is discussed. In Section 7 a duality picture of the G-L theory of the FQH states is developed. The results obtained in Sections 3, 4, 5 and 6 are rederived in the dual picture. The dual picture allows us to directly apply our results on the FQH states to the chiral spin states. In Section 8 we show that the ground degeneracy on a genus g Riemann surface is given by $\tilde{q}^g$ if the quasi-particle excitations have statistics $\theta = \frac{\pi\tilde{p}}{\tilde{q}}$. In Section 9 we discuss the concept of topological symmetry and conclude the paper.

II. GROUND STATE DEGENERACY AND ITS LIFTING BY IMPURITY POTENTIALS

In this section we discuss the ground state degeneracy of an FQH system on a torus geometry using elementary methods. We show how and to what extent the degeneracy is lifted by weak impurity potentials. This is done by projecting the impurity potentials onto the subspace of the ground states and by applying degenerate perturbation theory. This approach was first taken by Tao and Haldane.[16,17] Here we give a more detailed analysis. A very simple effective form of the impurity potentials is derived, from which the dependences of the impurity effects on the system size and the phases of the boundary conditions are clearly seen. In the end of the section, we remark on the practical significance of the degeneracy lifting.

We first give a brief review of the magnetic translation group. Consider an electron of charge $-e$ on a rectangular plane of size $L_1 \times L_2$, with a magnetic field B in the perpendicular direction ($\hat{z}$). In the absence of impurities, the Hamiltonian is

$$H = \frac{1}{2m}\left\{\left(-i\hbar\frac{\partial}{\partial x} + eA_x\right)^2 + \left(-i\hbar\frac{\partial}{\partial y} + eA_y\right)^2\right\}, \tag{2.1}$$

where (A_x, A_y) is the vector potential such that

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = B \tag{2.2}$$

The Hamiltonian has a symmetry of magnetic translations

$$t(\vec{a}) = e^{i\vec{a}\cdot\vec{k}/\hbar} \tag{2.3}$$

where $\vec{a}$ is a vector in the plane, and $\vec{k}$ is an operator (pseudo momentum) defined by

$$\begin{cases} k_x = -i\hbar\frac{\partial}{\partial x} + eA_x + eBy \\ k_y = -i\hbar\frac{\partial}{\partial y} + eA_y - eBx \end{cases} \tag{2.4}$$

It can be easily shown that $\vec{k}$ (and therefore $t(\vec{a})$) commutes with the dynamical momenta:

$$\begin{cases} \Pi_x = -i\hbar\frac{\partial}{\partial x} + eA_x \\ \Pi_y = -i\hbar\frac{\partial}{\partial y} + eA_y \end{cases} \tag{2.5}$$

and therefore with the Hamiltonian (2.1). From the commutator

$$[k_x, k_y] = i\hbar eB, \tag{2.6}$$

we have

$$t(\vec{a})t(\vec{b}) = t(\vec{b})t(\vec{a}) \cdot e^{-i(\vec{a}\times\vec{b})/l^2}, \tag{2.7}$$

where $l \equiv \left(\frac{\hbar}{eB}\right)^{\frac{1}{2}}$ is the magnetic length, and $\vec{a} \times \vec{b}$ means $\hat{z} \cdot (\vec{a} \times \vec{b})$.

When there are N_e electrons, each with a kinetic energy of (2.1), interacting mutually via a potential

$$V(\vec{r} - \vec{r}'), \tag{2.8}$$

the many-body magnetic translation

$$T(\vec{a}) \equiv \prod_{j=1}^{N_e} t_j(\vec{a}) \tag{2.9}$$

leaves the Hamiltonian of the system invariant, where t_j acts on the jth electron. In order to utilize this symmetry, we impose on the many-body wave funciton the periodic boundary conditions:

$$\begin{cases} t_j(\vec{L}_1)\psi = \psi \\ t_j(\vec{L}_2)\psi = \psi \end{cases} \tag{2.10}$$

where $\vec{L}_1 = L_1\vec{x}$, $\vec{L}_2 = L_2\vec{y}$. This means that the wave function is the same when an electron is magnetically translated $\vec{L}_1$ or $\vec{L}_2$ across the plane.

We assume there are N_s (integer) magnetic flux quanta through the surface,

$$N_s = \frac{L_1 L_2}{2\pi l^2}, \tag{2.11}$$

which is also the total number of single particle states in a Landau level. Corresponding to a fractional filling of the lowest Landau level, we have

$$N_e = \frac{p}{q} N_s, \tag{2.12}$$

where p and q are mutually prime integers. The translations which also leave the boundary conditions (2.11) invariant are

$$\begin{cases} T_1 \equiv T(\vec{L}_1/N_s) \\ T_2 \equiv T(\vec{L}_2/N_s) \end{cases}, \tag{2.13}$$

and their integral powers. We can thus choose a ground state ψ_0 to be an eigenstate of T_2, *i.e.*,

$$T_2\psi_0 = e^{i\lambda}\psi_0 \tag{2.14}$$

where λ is a real number because of the unitarity of T_2. Moreover, since

$$\begin{aligned} T_1T_2 &= T_2T_1e^{-iN_e(\vec{L}_1\times\vec{L}_2)/(N_s^2l^2)} \\ &= T_2T_1e^{-i2\pi p/q}, \end{aligned} \tag{2.15}$$

there will be $q-1$ more states degenerate (in energy) with ψ_0. These are

$$\psi_n \equiv T_1^n\psi_0 \quad , \quad n = 1, 2, \ldots, q-1 \tag{2.16}$$

and they are eigenstates of T_2,

$$T_2\psi_n = e^{i\lambda}e^{i2\pi p_n/q}\psi_n \tag{2.17}$$

with different eigenvalues, and therefore they are orthogonal to ψ_0 and to one another. This implies that the ground states are at least q-fold degenerate.

In the remaining part of this section, we assume there are exactly q ground states. Then T_1^q and T_2^q are constants within the ground state subspace. We now consider a weak impurity potential

$$\begin{aligned} U &= \sum_j U(\vec{r}_j) \\ &= \sum_{\vec{k}} \bar{U}(\vec{k}) \sum_j e^{i\vec{k}\cdot\vec{r}_j} \end{aligned} \tag{2.18}$$

where $\vec{k} = \left(\frac{2\pi n_1}{L_1}, \frac{2\pi n_2}{L_2}\right)$ is a Fourier wave vector, with n_1, n_2 being integers. Since the many-body states are antisymmetric in the electron labels, we can effectively write

$$U = N_e \sum_{\vec{k}} \bar{U}(\vec{k})e^{i\vec{k}\cdot\vec{r}} \tag{2.19}$$

where $\vec{r}$ now stands for the coordinate of any one electron. We assume that the potential is weaker compared with the energy gap above the ground state energy, so that we can use first order degenerate perturbation within the ground state subspace.

The fact that the ground states form an irreducible representation of T_1 and T_2 implies a number of selection rules. Consider the commutation relations,

$$\begin{aligned} T_1^q e^{i\vec{k}\cdot\vec{r}} &= e^{i\vec{k}\cdot\vec{r}}T_1^q e^{i2\pi n_1 q/N_s} \\ T_2^q e^{i\vec{k}\cdot\vec{r}} &= e^{i\vec{k}\cdot\vec{r}}T_2^q e^{i2\pi n_2 q/N_s}. \end{aligned} \tag{2.20}$$

Taking the matrix elements of both sides of these equations, we have

$$\begin{aligned} \left\langle \psi_{n'}|e^{i\vec{k}\cdot\vec{r}}|\psi_n \right\rangle &= \left\langle \psi_{n'}|e^{i\vec{k}\cdot\vec{r}}|\psi_n \right\rangle e^{i2\pi n_1 q/N_s} \\ \left\langle \psi_{n'}|e^{i\vec{k}\cdot\vec{r}}|\psi_n \right\rangle &= \left\langle \psi_{n'}|e^{i\vec{k}\cdot\vec{r}}|\psi_n \right\rangle e^{i2\pi n_2 q/N_s}, \end{aligned} \tag{2.21}$$

where we have used the fact that T_1^q and T_2^q are effectively constants. The matrix element is zero unless

$$\begin{aligned} n_1 &= l_1 N_s/q \\ n_2 &= l_2 N_s/q, \end{aligned} \tag{2.22}$$

where l_1, l_2 are integers. The impurity potential can therefore be written effectively as

$$U = N_e \sum_{l_1 l_2} \bar{U}(l_1 K_1, l_2 K_2) e^{i(l_1 K_1 x + l_2 K_2 y)} \tag{2.23}$$

where $K_1 = L_2/(ql^2), K_2 = L_1/(ql^2)$, and they are proportional to the system size. Thus, if the potential is smooth within a linear scale of much larger than ql^2/L_1 and ql^2/L_2, then $\bar{U}(l_1 K_1, l_2 K_2)$ will be exponentially small (except for $(l_1, l_2) = (0,0)$). Furthermore, if the ground states are made of primarily the single particle states in the lowest Landau levels, then, as has been shown in Ref. 16, we can write (2.23) effectively as

$$U = N_e \sum_{l_1 l_2} \bar{U}(l_1 K_1, l_2 K_2) e^{-\frac{1}{2}\left[\left(\frac{l_1 L_2}{ql}\right)^2 + \left(\frac{l_2 L_1}{ql}\right)^2\right]} t\left(\frac{l_2}{q}\vec{L}_1 - \frac{l_1}{q}\vec{L}_2\right), \tag{2.24}$$

where t is a magnetic translation acting on an electron.

The extra Gaussian factor makes the potential extremely small, even if the potential is not smooth. To lowest orders of $e^{-\frac{1}{2}\left(\frac{L_2}{ql}\right)^2}$ or $e^{-\frac{1}{2}\left(\frac{L_1}{ql}\right)^2}$, we can write

$$\begin{aligned} U = {} & u_1 t\left(\frac{\vec{L}_1}{q}\right) + U_1^* t\left(\frac{-\vec{L}_1}{q}\right) \\ & + u_2 t\left(\frac{\vec{L}_2}{q}\right) + U_2^* t\left(\frac{-\vec{L}_2}{q}\right) \end{aligned} \tag{2.25}$$

where $u_1 = N_e e^{-\frac{1}{2}\left(\frac{L_1}{ql}\right)^2} \bar{u}(0, K_2)$, $u_2 = N_e e^{-\frac{1}{2}\left(\frac{L^2}{ql}\right)^2} \bar{u}(-K_1, 0)$, and we have also ignored the constant part $N_e \bar{u}(0,0)$.

We now proceed to derive the effective forms of $t\left(\frac{\vec{L}_1}{q}\right)$ and $t\left(\frac{\vec{L}_2}{q}\right)$. Define an integer r satisfying

$$+pr + qm = 1 \quad , \quad |r| < \frac{q}{2} \quad , \quad m = \text{integer}, \tag{2.26}$$

which has unique solution, if q is odd. Then it can be shown that

$$T_1^{-r} t\left(\frac{\vec{L}_1}{q}\right) \qquad \text{and} \qquad T_2^{-r} t\left(\frac{\vec{L}_2}{q}\right) \tag{2.27}$$

commute with both T_1 and T_2. They must be constants within the subspace of the ground states which form an irreducible representation of the group generated by T_1 and T_2. We can thus write

$$t\left(\frac{\vec{L}_1}{q}\right) = e^{i\phi_1} T_1^r$$
$$t\left(\frac{\vec{L}_2}{q}\right) = e^{i\phi_2} T_2^r \tag{2.28}$$

The potential (2.25) can then be written effectively as

$$U = \bar{U}_1 T_1^r + \bar{U}_1^* T_1^{-r} + \bar{U}_2 T_2^r + \bar{U}_2^* T_2^{-r} \tag{2.29}$$

where $\bar{U}_1 = U_1 e^{i\phi_1}$ and $\bar{U}_2 = U_2 e^{i\phi_2}$.

Next, we consider the effect of changing the boundary condition (2.10) to

$$t_j(\vec{L})\psi(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{L}}\psi(\vec{\alpha}). \tag{2.30}$$

The states satisfying different boundary conditions can be connected by the twister $b(\vec{\alpha})$ defined as

$$b(\vec{\alpha}) = T(\vec{\alpha} \times \hat{z} l^2). \tag{2.31}$$

If $\psi(0)$ is an eigenenergy state satisfying (2.10), then

$$\psi(\vec{\alpha}) = b(\vec{\alpha})\psi(0) \tag{2.32}$$

is an eigenstate of the same energy (in the absence of impurity potentials) satisfying (2.30). It must be kept in mind, that (2.32) is not a gauge transformation unless the operators are also transformed accordingly. In other words, (2.32) is a change of boundary condition, if the operators of observables remain unchanged.

We can of course, keep the wave function unchanged, but transform the operators by $b(\vec{\alpha})$. Then we have

$$U = \bar{U}_1 e^{i\alpha_1 L_1 rp/q} T_1^r + \bar{U}_2 e^{i\alpha_2 L_2 rp/q} T_2^r + (h.c.) \tag{2.33}$$

The Schrödinger equation becomes:

$$\begin{cases} \epsilon\psi_n = \bar{U}_1 e^{i\alpha_1 L_1 rp/q}\psi_{n+r} + \bar{U}_1^* e^{-i\alpha_1 L_1 rp/q}\psi_{n-r} \\ \qquad + \left(\bar{U}_2 e^{i\alpha_2 L_2 rp/q} e^{ir\lambda} e^{i2\pi prn/q} + c.c.\right)\psi_n \\ \psi_{n+q} = e^{i\delta}\psi_n \end{cases} \tag{2.34}$$

where $e^{i\delta}$ is the constant of T_1^q. This can be transformed to the standard Harper's equations

$$\begin{cases} \epsilon\phi_n = R_1(\phi_{n+1} + \phi_{n-1}) \\ \qquad + 2R_2 \cos\left(\frac{2\pi r}{q} n + \frac{\alpha_2 L_2 rp}{q} + r\lambda + \theta_2\right)\phi_n \\ \phi_{n+q} = e^{i[\theta_1 q + \alpha_1 L_1 rp + r\delta]}\phi_n, \end{cases} \tag{2.35}$$

where $R_1 e^{i\theta_1} = \bar{U}_1,\ R_2 e^{i\theta_2} = \bar{U}_2$, and $\phi_n = e^{-i[\theta_1+\alpha_1 L_1 rp/q]n}\psi_{nr}$.

The band structure of (2.35) is well known. There are q bands, $\epsilon_j(\alpha_1, \alpha_2)$, with a periodicity of $\alpha_1 = \frac{2\pi}{rpL_1}$ and $\alpha_2 = \frac{2\pi}{rpL_2}$. This periodicity implies that there are $(rp)^2$ inequivalent boundary conditions giving rise to the same ground state energy splittings. For a large aspect ratio ($L_1 \ll L_2$), we have $R_2 \ll R_1$. The band widths are of order R_1/q, and the gaps are about R_2. In the other limit ($L_1 \gg L_2$), the roles of R_1 and R_2 are exchanged.

In any case, there is a unique ground state for each (α_1, α_2). It is then tempting to conclude that the Hall conductance should be an integer using the theory of topological invariant. This conclusion is wrong for two reasons. First, the arguments of topological invariant are only applicable to a state separated from others by energy gaps which do not become zero in the thermodynamic limit.[13] Secondly, linear response theory does not apply when the gaps are small such that Zener tunneling becomes important. However, both the linear response theory and the topological invariant arguments are valid for a group of states which are separated from others by finite energy gaps, even though the energy gaps among themselves are infinitesimal. At a temperature larger than the energy splittings, the q states are equally populated. The total Hall conductance can be calculated as the average of the contribution from each state[18] as if the Kubo formula is applicable to each of them. A more direct way is to invoke the topological invariance, and to calculate it in the absence of the impurity potential. Both methods should, of couse, give the same result: $\sigma_H = \frac{p}{q}\frac{e^2}{h}$.

III. GROUND STATE DEGENERACY OF THE FQH STATES – AN EFFECTIVE THEORY APPROACH

In this section we are going to give a simple heuristic argument about the ground state degeneracy of the FQH states. The approach is based on the effective Ginzburg-Landau (G-L) theory of the FQH effects.[19,20] More rigorous proof will be given in the next section. We will first consider the case with translation symmetry and reobtain the results in Ref. 10 and in the previous section.

The G-L theory for the FQH states can be written as

$$\begin{aligned}\mathcal{L}_{GL} =& \left[\phi^*(i\partial_0 - a_0 - eA_0)\phi - \frac{1}{2m}\phi^*(i\partial_i - a_i - eA_i)^2\phi + \mu|\phi|^2 - \lambda|\phi|^4\right] \\ &+ \left[\frac{-1}{4\pi q}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\right] + \dots \\ &= \mathcal{L}_\phi + \mathcal{L}_a + \dots\end{aligned} \tag{3.1}$$

where q is an odd integer and $f_{\mu\nu}$ is the field strength of a_μ. A_μ is the electromagnetic field and a_μ is an $U(1)$ gauge field introduced in Ref. 20. In this paper we will always regard A_μ as a fixed classical background field. We will not discuss the dynamics of the electromagnetic field.

The precise meaning of the G-L effective theory (3.1) is the following. An interacting (spinless) electron system in presence of electromagnetic field is described by the

Lagrangian

$$L_0 = \int d^2x \left[\psi^*(i\partial_0 - A_0)\psi - \frac{1}{2m_e}\psi^*(i\partial_i - eA_i)^2\psi\right]$$
$$- \int d^2x d^2x' |\psi(x)|^2 V(x-x')|\psi(x')|^2 \tag{3.2}$$

where $V(x-x')$ describes the interaction between electrons. After integrating out the electron field ψ we obtain an effective Lagrangian for the electromagnetic field A_μ:

$$e^{i\int dt L_{\text{eff}}(A_\mu)} = \int D\psi^* D\psi e^{i\int dt L_0(\psi, A_\mu)} \tag{3.3}$$

We say (3.1) is an effective theory of the electron system (3.2) if the same effective Lagrangian $L_{\text{eff}}(A_\mu)$ can be obtained after we integrate out ϕ and a_μ in (3.1):

$$e^{i\int dt L_{\text{eff}}(A_\mu)} = \int D\phi^* D\phi D a_\mu e^{i\int d^3x \mathcal{L}_{GL}(\phi, a_\mu, A_\mu)} \tag{3.4}$$

To satisfy (3.4) the G-L effective Lagrangian may be very complicated and contain high derivative terms. In (3.1) we only keep the lowest derivative terms because we are only interested in low energy and long wave length properties of the system.

All the physical properties of the electron system are measured by electromagnetic field A_μ. Therefore we may use the effective theory to study the physical properties of the effectron system. The G-L effective theories are useful because some states, like FQH states, have simple forms in terms of the effective theories. Some physical properties of those states are more transparent when expressed in therms of the effective theories. The effective theory approach is more general. It may apply to high genus Riemann surfaces where the ordinary magnetic translations can not be defined and used to study the ground state degeneracy. It also applies to the case with spatial dependent magnetic field.

Certainly the effective theories are not unique. Different ground states have simple forms only in the different effective theories. To study different FQH states we may use different effective theories to simplify the problem. The equivalence between (3.1) and (3.2) has not been proven in the sense that (3.4) is satisfied. However, it is demonstrated that (3.1) reproduces all known long distance and low energy properties of the FQH state. Therefore we will assume (3.1) is the effective theory of the FQH states, at least in the low energy long wave length limit. The effective theory approach is less rigorous because (3.4) has not been rigorously established yet.

According to Ref. 20, the FQH state is given by a mean field vacuum of (3.1):

$$\langle\phi\rangle = \sqrt{n} = \sqrt{\frac{e^2 B}{2\pi q}}$$
$$eA_\mu + a_\mu = 0 \tag{3.5}$$

where n is the electron density. The filling factor is given by $\nu = 1/q$ and the Hall conductance $\sigma_H = \frac{1}{q}\frac{e^2}{h}$.

Now let us consider the FQH state on a torus of size $L_1 \times L_2$. Notice that all the local quasi-particle fluctuations around the mean field vacuum, $\langle\phi\rangle = \sqrt{n}$, have finite energy gaps. Therefore the vacuum degeneracy (excitations with zero energy) can only come from the global excitations. On torus we may separate the local and the global excitations by writing

$$a_i + eA_i = \frac{\theta_i(t)}{L_i} + \delta a_i(x,t) \tag{3.6}$$

where $\delta a_i(x,t)$ satisfies

$$\int \delta a_i(x,t) d^2x = 0.$$

δa_i corresponds to the local excitations and θ_i global excitations. The effective theory of the global excitations θ_i is obtained by integrating out $a_0, \delta a_i$ and ϕ:

$$e^{i\int dt L_{\text{eff}}(\theta_i)} = \int Da_0 D\delta a_i D\phi e^{i\int d^3x \mathcal{L}_{GL}(a_\mu,\phi)} \tag{3.7}$$

Substituting (3.6) into (3.1) we find that (3.7) can be rewritten as

$$e^{i\int dt L_{\text{eff}}(\theta_i)} = e^{i\int dt \frac{1}{4\pi q}(\theta_1\dot\theta_2 - \theta_2\dot\theta_1)} \int Da_0 D\delta a_i D\phi e^{i\int d^3x [\mathcal{L}_\phi(a_\mu,\phi)+\mathcal{L}_a(\delta a_\mu)]}. \tag{3.8}$$

$L_{\text{eff}}(\theta_i)$ can be shown to have the following form

$$\begin{aligned} L_{\text{eff}}(\theta_i) = &\frac{1}{4\pi q}(\theta_1\dot\theta_2 - \theta_2\dot\theta_1) + f_i(\theta_i)\dot\theta_i + \frac{1}{2}M(\dot\theta_1^2 + \dot\theta_2^2) \\ &- V_1(\theta_1,\theta_2) + (\text{higher derivative terms}) \end{aligned} \tag{3.9}$$

The first term in (3.9) comes from the Chern-Simons term. The second and the third terms comes from the quantum fluctuations of ϕ and δa_i. (See Fig. 1.) The potential term $V_1(\theta_i)$ is non-zero because ϕ field condenses. V_1 is of order $\frac{n}{m}$. Notice that ϕ carries unit charge of the a_μ gauge field and the path integral in (3.8) is invariant under the following transformations

$$\begin{aligned} \phi \to \phi' &= e^{-i2\pi\left(\frac{p_1x_1}{L_1}+\frac{p_2x_2}{L_2}\right)}\phi \\ (\theta_1,\theta_2) &\to (\theta_1 + 2\pi p_1, \theta_2 + 2\pi p_2) \end{aligned} \tag{3.10}$$

where p_1 and p_2 are integers such that ϕ' is a single valued function on the torus. In other words, (p_1,p_2) are the winding numbers of ϕ on the torus coordinated by (x_1,x_2). The symmetry (3.10) implies that the potential $V_1(\theta_i)$ and the function $f_i(\theta)$ are periodic functions

$$V_1(\theta_1 + 2\pi p_1, \theta_2 + 2\pi p_2) = V_1(\theta_1,\theta_2) \tag{3.11}$$

$$f_i(\theta + 2\pi p_1, \theta_2 + 2\pi p_2) = f_i(\theta_1,\theta_2).$$

The explicit form of $V_1(\theta_i)$ may be obtained in semi-classical approximation. In this approximation we assume $\delta a_i = 0$ (in this case $f_i = 0$). The integration of $a_0(x_\mu)$ imposes a constraint

$$|\phi|^2 = n = \frac{e^2 B}{2\pi q}$$

The integration of ϕ is truncated to a summation of stationary points given by

$$\phi_{p_1p_2}(x) = \sqrt{n}\; e^{-i2\pi\left(\frac{p_1x_1}{L_1}+\frac{p_2x_2}{L_2}\right)} \tag{3.12}$$

where p_1 and p_2 are integers. Now (3.8) becomes

$$e^{i\int dtL_{\text{eff}}(\theta_i)} = e^{i\int dt\frac{1}{4\pi q}(\theta_1\dot\theta_2-\theta_2\dot\theta_1)} \times \sum_{p_1p_2} e^{-i\int dt\frac{n}{2m}\left[\frac{L_2}{L_1}(\theta_1+2\pi p_1)^2+\frac{L_1}{L_2}(\theta_2+2\pi p_2)^2\right]}$$

We find that

$$V_1(\theta_1,\theta_2) = \frac{n}{2m}\left(\frac{L_2}{L_1}\theta_1^2+\frac{L_1}{L_2}\theta_2^2\right)\Bigg|_{-\pi\le\theta_1,\theta_2\le\pi} \tag{3.13}$$

For other values of θ_i, $V_1(\theta_i)$ is determined by (3.11).

We would like to emphasize that the specific forms of the potential $V_1(\theta_i)$ and the function $f_i(\theta_i)$ are not important in our discussions. Our discussions (followed below) only depend on the periodic condition (3.11). The periodic condition is a consequence of the charge *one* boson (ϕ field) condensation and is very robust. The periodic condition can be changed only through phase transitions in which, for example, the charge one boson condensation changes to charge N boson condensation.

Now we are ready to study the dynamics of the global excitation governed by (3.9). The Lagrangian in (3.9) effectively describes a charged particle moving in a θ-space. There is a periodic potential in the θ-space $V_1(\theta_i)$ with period 2π in both θ_1 and θ_2 direction. The first and the second terms in (3.9) implies that there is a "magnetic" field in the θ-space with $\frac{2\pi}{q}$ flux per plaquette. This system has been studied in detail.[21,18] The Hamiltonian is given by

$$H = -\frac{1}{2M}\left[\left(\frac{\partial}{\partial\theta_1}+i\tilde A_1\right)^2+\left(\frac{\partial}{\partial\theta_2}+i\tilde A_2\right)^2\right]+V_1(\theta_1,\theta_2) \tag{3.15}$$

with $\frac{\partial}{\partial\theta_1}\tilde A_2 - \frac{\partial}{\partial\theta_2}\tilde A_1 = \frac{1}{2\pi q}+\epsilon^{ij}\frac{\partial}{\partial\theta_i}f_j$. The ground state of (3.15) is found to be q fold degenerate. One way to prove this result is to notice that H in (3.15) commutes with the magnetic translations T_1 and T_2:

$$\begin{aligned} T_1&: \theta_1\to\theta_1+2\pi\\ T_2&: \theta_2\to\theta_2+2\pi. \end{aligned} \tag{3.16}$$

T_1 and T_2 satisfy an algebra

$$T_1T_2 = e^{i\frac{2\pi}{q}}T_2T_1 \tag{3.17}$$

whose irreducible representation has a dimension of q. The ground states of H must form a representation of (3.17) and hence have to be at least q fold degenerate. Sometimes the ground states may be nq fold degenerate, if different irreducible representations of (3.17) *happen* to have the same lowest energy. However, this is not a generic situation.

We would like to remark that the magnetic translations T_1 and T_2 are the quantum realization of the classical transformations in (3.10) with (m,n) equal to (1,0) and (0,1) respectively. In absence of the Chern-Simons term the transformations (3.10) should be regarded as the gauge symmetry. This means that we should identify θ_1 with $\theta_1+2\pi$ and θ_2 with $\theta_2+2\pi$. The θ-space is actually *finite*. However, in presence of the Chern-Simons term the quasi-particles (and the quasi-holes) carry fractional charge. The quasi-particle-quasi-hole tunneling process described in Section 5 produces physical operators proportional to T_1 or T_2. Noticing that T_1 and T_2 do not commute, we can not regard T_1 and T_2 as the gauge transformations, because the gauge transformations should leave all physical operators invariant. Noticing that T_1^q and T_2^q commute with T_1 and T_2 (see (3.17)), we can still regard T_1^q and T_2^q as gauge transformations. This implies that we may identify θ_1 with $\theta_1 + 2\pi q$ and θ_2 with $\theta_2 + 2\pi q$. The θ-space is still finite.

We would like to emphasize that the above result does not depend on the particular simple form of the approximated Hamiltonian (3.15). The ground states remain to be q fold degenerate as long as there exists the magnetic translations which satisfy the algebra (3.17) and commute with the Hamiltonian (3.15). This only requires that the physics described by (3.15) is periodic and there is $\frac{2\pi}{q}$ flux in an area of period square. Our result holds even for the following general Hamiltonian

$$\begin{aligned} H = K &\left(i \left(\frac{\partial}{\partial\theta_1} + i\tilde{A}_1 \right), i \left(\frac{\partial}{\partial\theta_2} + i\tilde{A}_2 \right) \right) \\ &+ V(\theta_1, \theta_2) \end{aligned} \tag{3.18}$$

where $K(x, y)$ is an arbitrary positive function and $V(\theta_1, \theta_2)$ an arbitrary periodic potential satisfying (3.11). The "magnetic" field $\tilde{B}(\theta_i) = \frac{\partial}{\partial\theta_1}\tilde{A}_2 - \frac{\partial}{\partial\theta_2}\tilde{A}_1$ is periodic with period 2π in both θ_1 and θ_2. $\tilde{B}(\theta_i)$ further satisfies

$$\int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \tilde{B}(\theta_i) = \frac{2\pi}{q} \tag{3.19}$$

The periodicity in θ_1 and θ_2 is a consequence of the gauge symmetry (3.10) and the $\frac{2\pi}{q}$ flux is determined by the coefficient of the Chern-Simons term. Therefore we expect that the Hamiltonian for θ_1 and θ_2 has a form of (3.18) even if we include all the quantum corrections (except for a non-perturbative effect which vanishes exponentially in thermodynamic limit. See Section 5.)

Now let us include impurity potential to our system. In the framework of the effective theory, the effects of impurity potential may be included by allowing the various coefficients in the effective G-L theory to have a spatial dependence, except the coefficient in front of the Chern-Simons term which must be a constant as required by the gauge symmetry. We also allow the magnetic field B to have a spatial dependence. In this general situation, the above discussions are still valid. The transformations (3.10) remain to be a symmetry of the path integral in (3.8) and the Hamiltonian for θ_1 and θ_2 still takes the form in (3.18). Thus the ground states remain to be q fold degenerate.

We would like to stress that the derivation presented in this section is not strictly correct. To obtain the effective theory for θ_i we have assumed that θ_i are slow variables. But the θ_i and other local fluctuations actually have similar energy scale. The separation of the global and the local fluctuations is quite artificial in this case. However, the effective theory of θ_i does contain correct algebraic structure. This is the reason why we obtain the correct result. A more rigorous and abstract derivation will be presented in the next section.

IV. TRANSFORMATION ALGEBRA ANOMALY AND THE GROUND STATE DEGENERACY OF THE FQH STATES

In this section we are going to give a general proof of the ground state degeneracy of the FQH states on a torus. We will construct operators similar to, but more general than, the magnetic translation operators introduced in Section 2. These operators commute with the Hamiltonian of the system, but not with each other, implying the degeneracy of the energy eigenstates of the systems. The proof given here is general enough to apply to situations with random potential, spatial dependent magnetic field and many other perturbations, as long as all quasi-particle excitations have finite energy gaps.

The essence of the approach in the last section to the ground state degeneracy is the magnetic translations (3.16) (in θ-space), which is nothing but the gauge transformations (3.10). Therefore the better approach is to directly use the algebra of the gauge transformations (3.10) to calculate the ground state degeneracy without deriving the effective Lagrangian (3.9) for the global excitations. To do so we first need to quantize the Lagrangian (3.1). In the following we will allow μ, m, λ and the magnetic field in (3.1) to have a spatial dependence.

We may quantize the gauge field[22,23] a_μ in the gauge

$$a_0 = 0 \tag{4.1}$$

The equation of motion for a_0 serves as a constraint:

$$\begin{aligned} G[f] &= i \int d^2x f(x) \frac{\delta L_{\text{eff}}}{\delta a_0} \\ &= i \int d^2x f(x) \left(-\phi^* \phi - \frac{1}{4\pi q} \epsilon^{ij} f_{ij} \right) \\ &= i \int d^2x f(x) G(x) = 0 \end{aligned} \tag{4.2}$$

where $f(x)$ is an arbitrary globally defined real function on the torus. After quantization the constraint (4.2) is met by demanding all the states in the physical Hilbert space to satisfy (the Gauss law)

$$\hat{G}[f] |\Psi_{\text{phy}}\rangle = 0. \tag{4.3}$$

From (3.1) we see that a_1 and a_2 canonically conjugate to each other

$$[\hat{a}_1(x), \hat{a}_2(y)] = i 2\pi q \delta^2(x-y). \tag{4.4}$$

Similarly,

$$[\hat{\phi}^\dagger(x), \hat{\phi}(y)] = i \delta^2(x-y). \tag{4.5}$$

Using (4.4) and (4.5) one can easily check that $\hat{G}$ generates a gauge transformation

$$\begin{aligned} e^{-i\hat{G}[f]} \hat{a}_i e^{i\hat{G}[f]} &= \hat{a}_i + i\partial_i f \\ e^{-i\hat{G}[f]} \hat{\phi} e^{i\hat{G}[f]} &= e^{-if} \hat{\phi} \end{aligned} \tag{4.6}$$

On the other hand, (4.3) implies that $e^{-i\hat{G}[f]}|\Psi_{\text{phy}}\rangle = |\Psi_{\text{phy}}\rangle$, meaning that the physical state should be gauge invariant. Because $f(x)$ is single valued on the torus we will call $e^{i\hat{G}[f]}$ a local gauge transformation.

Using $\hat{G}(x)$ we can also construct so-called large gauge transformations. Consider the operators

$$\begin{aligned} T_1 &= e^{i\int d^2x f_1(x)\hat{G}(x)} \\ T_2 &= e^{i\int d^2x f_2(x)\hat{G}(x)} \end{aligned} \tag{4.7}$$

where $f_i(x)$ have a 2π jump along a loop in x_i direction which goes all the way around the torus (Fig. 2). One can check

$$\begin{aligned} T_j^{-1}\hat{a}_i T_j &= \hat{a}_i + \partial_i f_j(x) \\ T_j^{-1}\hat{\phi} T_j &= e^{-if_j(x)}\hat{\phi} \end{aligned} \tag{4.8}$$

Notice that $\partial_i f_j(x)$ and $e^{if_j(x)}$ are smooth functions. T_j generate non-singular transformations and are well defined operators. From (4.4) and (4.5) T_1 and T_2 can be shown to satisfy the famous algebra

$$T_1 T_2 = e^{i\frac{2\pi}{q}} T_2 T_1 \tag{4.9}$$

The algebra (4.9) is very important. The non-communitivity of T_1 and T_2 is purely a quantum effect. Classically the transformations generated by f_1 and f_2 definitely commute with each other. Due to the algebra (4.9), there is no state which is invariant under both T_1 and T_2. Despite the classical Lagrangian is invariant under the transformation (4.8), the quantum states can not be invariant under T_1 and T_2. We will call this phenomenon transformation algebra anomaly. T_1 and T_2 are physical operators in the sense that they are generated by the physical tunneling process discussed in Section 5. *The physical Hilbert space is defined as a representation of the physical operators.* In particular, the physical Hilbert space form a representation of the algebra (4.9). Because the gauge transformation must commute with all the physical operators, we can not regard T_1 and T_2 as gauge transformations. However, from (4.9) we find that T_1^q and T_2^q commute with T_1 and T_2. We may regard T_1^q and T_2^q as generators of (large) gauge transformations. Since T_1^q and T_2^q commute we may require the physical states to be invariant under these large gauge transformations

$$\begin{aligned} T_1^q|\Psi_{\text{phy}}\rangle &= |\Psi_{\text{phy}}\rangle \\ T_2^q|\Psi_{\text{phy}}\rangle &= |\Psi_{\text{phy}}\rangle \end{aligned} \tag{4.10}$$

Because T_1 and T_2 commute with T_1^q and T_2^q, T_1 and T_2 naturally act on the physical Hilbert space, *i.e.,* a physical state when acted by T_1 and T_2 still remains to be a physical state.

The Hamiltonian of the system (3.1) is given by

$$\begin{aligned} H = &-\frac{1}{2}\hat{\phi}^\dagger(\partial_i + i\hat{a}_i + ieA_i)\frac{1}{m(x)}(\partial_i + ia_i + ieA_i)\hat{\phi} - \mu(x)\hat{\phi}^\dagger\hat{\phi} \\ &+ \lambda(x)(\hat{\phi}^\dagger\hat{\phi})^2 \end{aligned} \tag{4.11}$$

The Hamiltonian commutes with $\hat{G}(f)$, T_1^q and T_2^q. Therefore H acts on the physical Hilbert space. H also commutes with T_1 and T_2. Therefore each energy level of H is q fold degenerate and forms an irreducible representation of (4.9). In particular the ground states of the FQH state are at least q fold degenerate on the torus.

We would like to remark that once written in the G-L form, the system has a degeneracy, despite the disorders in the coefficients (A_i, μ, λ, m) of the theory as has been proven in Section 3 and this section. However, the reader should be warned that the effect of disorder in the original theory is a different story and is discussed in Section 2 and 5. Disorder in the original theory may give rise, in addition to the disorder in the G-L theory, to corrections to the G-L Hamiltonian (4.11), which breaks the symmetries of T_1 and T_2.

In the following we would like to discuss the ground state wave functions of the FQH state. In the boson ϕ condensed phase, the Hilbert space is divided into sectors. The states in each sector describe the quantum fluctuations around the stationary point

$$\begin{aligned} \phi &= \phi_{p_1 p_2} \\ a_i &= -eA_i + p_i \frac{2\pi}{L_i} \end{aligned} \tag{4.12}$$

where $\phi_{p_1p_2}$ is given by (3.12). Thus different sectors are labeled by two integers (p_1, p_2). The states in the sector (p_1, p_2) are given by the wave functional

$$\Psi[a_i, \phi] = \Psi\left[-eA_i + \frac{2\pi p_i}{L_i} + \delta a_i, \phi_{p_1p_2} + \delta\phi\right] \tag{4.13}$$

where δa_i and $\delta\phi$ are small fluctuations around the stationary point. In the thermodynamic limit the states in the different sectors do not mix, *i.e.,* the quantum fluctuations can not connect a state in one sector to another state in a different sector. The Hamiltonian does not contain off-diagonal terms (in the thermodynamic limit) which mix two different sectors. Let $|p_1, p_2\rangle$ denote the lowest energy state (the ground state) in the sector (p_1, p_2). Notice that T_1 and T_2 map a state in one sector to a state in a different sector. Since T_1 and T_2 commute with the Hamiltonian, they map the ground state of one sector to the ground state of another sector:

$$\begin{aligned} T_1|p_1, p_2\rangle &= e^{i\alpha(p_1,p_2)}|p_1+1, p_2\rangle \\ T_2|p_1, p_2\rangle &= e^{i\beta(p_1,p_2)}|p_1 p_2+1\rangle \end{aligned} \tag{4.14}$$

The ground states in different sectors all have the same energy.

In the above discussion we did not consider the gauge symmetries. The ground states in different sectors in general are not the physical states, *i.e.,* they do not satisfy (4.3) and (4.10). Only some particular superpositions of those ground states correspond to the physical ground states. In absence of the Chern-Simons term, all physical operators commute with T_1 and T_2. Since T_1 and T_2 commute we may require the physical states to be invariant under T_1 and T_2:

$$\begin{aligned} T_1|\Psi_{\text{phy}}\rangle &= |\Psi_{\text{phy}}\rangle \\ T_2|\Psi_{\text{phy}}\rangle &= |\Psi_{\text{phy}}\rangle \end{aligned} \tag{4.15}$$

We may choose the phase of $|p_1p_2\rangle$ such that $\alpha(p_1p_2)$ and $\beta(p_1p_2)$ in (4.14) are equal to zero. This is possible because T_1 and T_2 commute. It is not difficult to see that using $|p_1p_2\rangle$ we can only construct one physical ground state satisfying (4.15):

$$|0_{\text{phy}}\rangle = \sum_{p_1p_2} |p_1p_2\rangle \tag{4.16}$$

Therefore in absence of the Chern-Simons term the ground state is non-degenerate as we expected.

In presence of the Chern-Simons term T_1 and T_2 are the physical operators. The physical ground states form a representation of (4.9). In terms of $|p_1 p_2\rangle$, the q physical ground states satisfying (4.10) and forming a representation of (4.9) are given by

$$|n_{\text{phy}}\rangle = \sum_{p_1 p_2} e^{i\frac{2\pi}{q} n p_2} |p_1 p_2\rangle \tag{4.17}$$

where $n = 1, \ldots, q$. The phases of $|p_1 p_2\rangle$ have been chosen such that

$$\begin{aligned} \alpha(p_1 p_2) &= \frac{2\pi}{q} p_2 \\ \beta(p_1 p_2) &= 0 \end{aligned} \tag{4.18}$$

One can check that this choice of the phases is consistent with the algebra (4.9). The states $|n_{\text{phy}}\rangle$ satisfy

$$\begin{aligned} T_1 |n_{\text{phy}}\rangle &= |(n+1)_{\text{phy}}\rangle \\ T_2 |n_{\text{phy}}\rangle &= e^{-i\frac{2\pi}{q} n} |n_{\text{phy}}\rangle \\ |n_{\text{phy}}\rangle &= |(n+q)_{\text{phy}}\rangle \end{aligned} \tag{4.19}$$

In thermodynamic limit the perturbative Hamiltonian does not mix different physical ground states, and all the physical ground states have the same energy. However, for finite system the tunnel process described in Section 5 induces a term in the Hamiltonian which mixes the different ground states and lifts the ground state degeneracy.

V. ENERGY SPLIT OF THE GROUND STATES OF FINITE SYSTEM

In Section IV we show that the ground states of the FQH state are q fold degenerate even in presence of random potentials. This is because the generators of the algebra (4.9) commute with the Hamiltonian of the system. However, the above result is only valid in the thermodynamic limit. For systems with finite size there is a non-perturbative effect. After including the non-perturbative effect, the Hamiltonian obtains a small correction proportional to $\gamma e^{-L\sqrt{m^* \Delta}}$ which does not commute with T_1 and T_2. Therefore the energy of the ground states can be shown to have a split of order $\gamma e^{-L\sqrt{m^* \Delta}}$.

The non-perturbative effect comes from the following tunneling process. A pair of quasi-particles and quasi-holes is virtually created at a time t_0. The quasi-particle and quasi-hole move in opposite directions and propagate all the way around the torus. When they meet on the opposite side of the torus, they annihilate at time $t_0 + T$. The resulting new ground state is different from the old ground state. The magnitude of the tunneling amplitude is given by

$$\begin{aligned} |A| &= e^{-S} \\ S &= T\Delta + 2\frac{1}{2} m^* \left(\frac{L}{2T}\right)^2 T \end{aligned} \tag{5.1}$$

where Δ is the energy gap of the quasi-particle quasi-hole pair creation, m^* is the effective mass of the quasi-particle and L is the size of the torus. (In this section we will assume $L_1 = L_2 = L$.) S is minimized at $T = \frac{1}{2}L\sqrt{\frac{m^*}{\Delta}}$ with the minimum value $S = L\sqrt{\Delta m^*}$. Hence

$$|A| = e^{-L\sqrt{\Delta m^*}}. \tag{5.2}$$

Therefore the magnitude of the non-perturbative correction is exponentially small.

In order to obtain the explicit form of the non-perturbative corrections and to show that the corrections do not commute with T_1 and T_2, we need to study the tunneling process in more detail.

The quasi-particle in the FQH state is given by a vortex in ϕ field. The ansatz of the quasi-particle may be chosen to be

$$\begin{aligned}\frac{\phi(z)}{\sqrt{n}} &= \frac{z - z_0(t)}{|z - z_0(t)| + \xi} \\ a_0 &= 0 \\ a_1(z) + ia_2(z) &= \frac{i(z - z_0)}{|z - z_0|^2 + \xi'^2} - eA_1 - ieA_2\end{aligned} \tag{5.3}$$

where $z = x_1 + ix_2$ and z_0 is the position of the quasi-particle. ξ and ξ' in (5.3) are positive which determine the size of the quasi-particle. A pair of the quasi-particle and quasi-hole is described by the ansatz

$$\begin{aligned}\frac{\phi(z)}{\sqrt{n}} &= \frac{(z - z_0(t))(z - \tilde{z}_0^*(t)) + f^2(|z_0 - \tilde{z}_0|/\xi)}{|z - z_0(t)||z - \tilde{z}_0(t)| + \xi^2} \\ a_1(z) + ia_2(z) &= \frac{i(z - z_0)}{|z - z_0|^2 + \xi'^2} - \frac{i(z - \tilde{z}_0)}{|z - \tilde{z}_0|^2 + \xi'^2} - eA_1 - ieA_2\end{aligned} \tag{5.4}$$

where z_0 and $\tilde{z}_0$ are the positions of the quasi-particle and the quasi-hole respectively. The function $f(x)$ in (5.4) satisfies $f(0) = \xi$ and $f(x) = 0|_{x>1}$. When $|z_0 - \tilde{z}_0|$ is large, (5.4) describes a vortex and an anti-vortex. When $z_0 - \tilde{z}_0 = 0$, (5.4) describes a mean field vacuum state. Thus by separating z_0 and $\tilde{z}_0$, (5.4) describes a process of creation of a quasi-particle and a quasi-hole.

In order to construct the quasi-particle and the quasi-hole on the torus, ϕ and a_i must satisfy the periodic boundary conditions. We find that on the torus a pair of the quasi-particle and the quasi-hole is given by

$$\begin{aligned}\frac{\phi(z)}{\sqrt{n}} &= \frac{F(z|z_0, \tilde{z}_0)L + f(z_0, \tilde{z}_0)e^{-i\frac{2\pi}{L^2}Re(z_0 - \tilde{z}_0)Im\ z}}{|F(z|z_0, \tilde{z}_1)| + \xi} \\ a_1(z) + ia_2(z) &= \sum_{mn}\left(\frac{i(z - z_0 - Z_{mn})}{|z - z_0 - Z_{mn}|^2 + \xi'^2} - \frac{i(z - \tilde{z}_0 - Z_{mn})}{|z - \tilde{z}_0 - Z_{mn}|^2 + \xi'^2}\right) - eA_1 - ieA_2\end{aligned} \tag{5.5}$$

where $Z_{mn} = mL + inL$. $f(z_0, \tilde{z}_0)$ is a positive periodic function of z_0 and $\tilde{z}_0$.

$$f(z_0 + mL + inL, \tilde{z}_0 + \tilde{m}L + i\tilde{n}L) = f(z_0, \tilde{z}_0). \tag{5.6}$$

$f(z_0, \tilde{z}_0)$ is non-zero only when $|z_0 - \tilde{z}_0 - Z_{mn}| < \xi$ for some m and n. $f(z_0, \tilde{z}_0) = \xi$. $F(z|z_0, \tilde{z}_0)$ in (5.8) is a periodic function in z:

$$\begin{aligned} &F(z + L|z_0, \tilde{z}_0) \\ &= F(z + iL|z_0, \tilde{z}_0) \\ &= F(z|z_0, \tilde{z}_0) \end{aligned} \tag{5.7}$$

F has a zero at z_0

$$F(z|z_0, \tilde{z}_0) \sim (z - z_0)|_{z \to z_0} \tag{5.8}$$

and an "anti-zero" at $\tilde{z}_0$

$$F(z|z_0, \tilde{z}_0) \sim (z^* - \tilde{z}_0^*)|_{z \to \tilde{z}_0} \tag{5.9}$$

An order $O(1)$ function F satisfying (5.7)–(5.9) is given by

$$\begin{aligned} F(z|z_0, \tilde{z}_0) =& e^{-i\frac{\pi}{L^2}\mathrm{Re}(z_0 - \tilde{z}_0)[\mathrm{Im}\ 2z + L]} e^{-\frac{\pi}{L^2}[\mathrm{Im}(z - z_0)]^2 - \frac{\pi}{L^2}[\mathrm{Im}(z - \tilde{z}_0)]} \\ & \theta_1\left(\frac{z - z_0}{L}|i\right) \theta_1^*\left(\frac{z - \tilde{z}_0}{L}|i\right) \end{aligned} \tag{5.10}$$

where $\theta_1(u|\tau)$ is the odd elliptic theta function[24] satisfying

$$\begin{aligned} \frac{\theta_1(u + 1|\tau)}{\theta_1(u|\tau)} &= -1 \\ \frac{\theta_1(u + \tau|\tau)}{\theta_1(u|\tau)} &= -e^{-i\pi(2u + \tau)} \\ \theta_1(u|\tau) &\sim u|_{u \to 0} \end{aligned} \tag{5.11}$$

when $z_0 = \tilde{z}_0$ or $z_0 = \tilde{z}_0 + L$, F satisfies

$$\begin{aligned} F(z|z_0, z_0) &= |F(z|z_0, z_0)| \\ F(z|z_0, z_0 + L) &= e^{-i\frac{2\pi}{L}\mathrm{Im}z}|F(z|z_0, z_0)| \end{aligned} \tag{5.12}$$

When z_0 and $\tilde{z}_0$ are well separated, f in (5.5) can be dropped and (5.5) describes a quasi-particle at z_0 and a quasi-hole at $\tilde{z}_0$. When z_0 and $\tilde{z}_0$ are close to each other (5.5) describes creation or annihilation of the quasi-particle and the quasi-hole.

The tunneling process described at the beginning of this section is obtained by choosing $z_0(t)$ and $\tilde{z}_0(t)$ in (5.5) to be

$$\begin{aligned} z_0(t) = \tilde{z}_0(t) = 0 \quad &, \qquad t < t_0 \\ z_0(t) = -\tilde{z}_0(t) = \frac{L}{2T}t \quad &, \quad t_0 < t < t_0 + T \\ z_0(t) = -\tilde{z}_0(t) = \frac{L}{2} \quad &, \quad t > t_0 + T \end{aligned} \tag{5.13}$$

Before the tunneling ($t < t_0$) the vacuum state is given by (5.5) with $z_0 = \tilde{z}_0$:

$$\frac{\phi}{\sqrt{n}} = 1 \quad , \quad a_1 + ia_2 = -eA_1 - ieA_2 \tag{5.14}$$

After the tunneling ($t > t_0 + T$) we have $z_0 = -\tilde{z}_0 = \frac{L}{2}$ and the vacuum state is given by

$$\frac{\phi}{\sqrt{n}} = e^{-i\frac{2\pi}{L}x_2}$$
$$a_1 = -eA_1 \quad , \quad a_2 = \frac{2\pi}{L} - eA_2 \tag{5.15}$$

The two states (5.14) and (5.15) are related by the transformation T_2. This result is easy to understand because form (Fig. 3) one can see that the quasi-particle quasi-hole tunneling adds a unit a_μ flux quantum to the hole of the torus.

Let us use two integers (p_1, p_2) to label different mean field vacua

$$(p_1, p_2): \qquad \frac{\phi}{\sqrt{n}} = e^{-i\frac{2\pi}{L}(p_1x_1+p_2x_2)}$$
$$a_i = \frac{2\pi}{L}p_i - eA_i \tag{5.16}$$

The different vacua are connected by transformations T_1 and T_2. From (5.14) and (5.15) we see that the tunneling process for the quasi-particle moving x_1 direction changes the $(0,0)$ vacuum to the (0,1) vacuum. Similarly, the tunneling in x_2 direction changes the (0,0) vacuum to the (-1,0) vacuum. We may define the amplitudes of the above tunnelings, $(0,0) \to (0,1)$ and $(0,0) \to (-1,0)$ to have zero phase (*i.e.*, the amplitudes are real and positive). The tunneling from, say, (p_1, p_2) to $(p_1, p_2 + 1)$ can be obtained by making a gauge transformation. The configuration describes the tunneling $(p_1, p_2) \to (p_1, p_2 + 1)$ is given by

$$\phi' = e^{-i\frac{2\pi}{L}(p_1x_1+p_2x_2)}\phi$$
$$a_i' = a_i + \frac{2\pi}{L}\ p_i \tag{5.17}$$

where ϕ and a_i are given by (5.5). The phase of the tunneling amplitude is

$$\varphi_{(p_1,p_2)(p_1,p_2+1)} = \int d^3x\left[\mathcal{L}_{GL}(a_i', \varphi') - \mathcal{L}_{GL}(a_i, \varphi)\right]$$
$$= \int d^3x \frac{-1}{4\pi q}\left(\frac{2\pi p_1}{L}\partial_0 a_2 - \frac{2\pi p_2}{L}\partial_0 a_1\right) \tag{5.18}$$

where a_i is given by (5.5). After some calculations we find that

$$\varphi_{(p_1,p_2)(p_1,p_2+1)} = \frac{2\pi p_1}{2q} \tag{5.19}$$

Similarly, we find that for the tunneling $(p_1, p_2) \to (p_1 - 1, p_2)$:

$$\varphi_{(p_1,p_2)(p_1-1,p_2)} = \frac{2\pi p_2}{2q} \tag{5.20}$$

$\varphi_{(p_1,p_2)(p_1',p_2')}$ as the phase of the hopping amplitude between *two* different states is not a physically observable quantity. Physically observable quantities are the phases of tunneling with the same initial and final state. Let us consider the following tunneling process: $(p_1,p_2) \to (p_1,p_2+1) \to (p_1-1,p_2+1) \to (p_1-1,p_2) \to (p_1,p_2)$. (Fig. 4.) The total phase of the tunneling is given by

$$
\begin{aligned}
&\varphi_{(p_1,p_2)(p_1,p_2+1)} + \varphi_{(p_1,p_2+1)(p_1-1,p_2+1)} \\
&- \varphi_{(p_1-1,p_2)(p_1-1,p_2+1)} - \varphi_{(p_1,p_2)(p_1-1,p_2)} = \frac{2\pi}{q}
\end{aligned}
\tag{5.21}
$$

The tunneling in x_1 direction, $(p_1,p_2) \to (p_1,p_2+1)$, changes one ground state to another and defines a unitary matrix U_1 acting on the ground states. The tunneling in x_2 direction, $(p_1,p_2) \to (p_1-1,p_2)$, defines a unitary matrix U_2 acting on the ground states. The result (5.21) implies that U_1 and U_2 satisfy the algebra

$$U_2^{-1}U_1^{-1}U_2U_1 = e^{i\frac{2\pi}{q}} \tag{5.22}$$

which is identical to the algebra satisfied by T_1 and T_2. Noticing that U_1 (U_2) changes a state to its transformed state by $T_2(T_1)$, we may conclude that in the subspace spanned by the degenerate ground states, U_2 and U_1 are proportional to T_1 and T_2 respectively

$$
\begin{aligned}
U_1 &= \gamma_2 T_2 \\
U_2 &= \gamma_1 T_1
\end{aligned}
\tag{5.23}
$$

After including the non-perturbative effects, the Hamiltonian (4.11) receives a correction

$$
\begin{aligned}
\Delta H &= A(U_1 + U_1^\dagger + U_2 + U_2^\dagger) \\
&= A(\gamma_1 T_1 + \gamma_2 T_2 + h.c.) \\
A &= \gamma e^{-L\sqrt{\Delta m^*}}
\end{aligned}
\tag{5.24}
$$

ΔH does not commute with T_1 and T_2. The ground state degeneracy is lifted by the non-perturbative effects. The energy split is of order $\gamma e^{-L\sqrt{\Delta m^*}}$.

We would like to point out that the tunneling process described by Fig. 4 can be deformed into two linking loops (Fig. 5). Therefore the phase in (5.21) is equal to the phase we obtained by moving one quasi-particle around another. This phase is given by 2θ where θ is the statistical angle of the quasi-particle. Thus (5.22) can be rewritten as

$$U_2^{-1}U_1^{-1}U_2U_1 = e^{i2\theta}. \tag{5.25}$$

Because the ground states form a representation of the algebra (5.25), the ground state degeneracy is *directly* determined by the statistics of the quasi-particles.

We would like to remark that the tunnelings along two different tunneling paths given by, say, $x_2 = 0$ and $x_2 = \Delta x_2$ have a phase difference

$$\Delta\varphi = \frac{e^2 B}{q}\Delta x_2 L$$

because the quasi-particle carries the electrical charge $\frac{e}{q}$. Therefore, after summing up all the tunneling paths associated with different Δx_2 the factor γ in the tunneling amplitude (5.24) takes a form

$$\gamma \propto \int d\Delta x_2 e^{i\frac{e^2 B}{q}\Delta x_2 L} e^{-L\sqrt{m^*\Delta}}. \tag{5.26}$$

If our system respects translation symmetry m^* and Δ in (5.26) do not depend on Δx_2 and we find that $\gamma = 0$. The ground state degeneracy is exact even in systems with finite size. This agrees with the result in Ref. 10. Only when the translation symmetries are broken can the ground state degeneracy be really lifted by the non-perturbative effects.

Strictly speaking, the total tunneling amplitude is given by the sum of the amplitudes of all different tunneling path C:

$$A \propto \int D\vec{x}(t) e^{i\frac{e}{q}\oint_C d\vec{x}\cdot\vec{A}} \times e^{-\oint_C dt\left[\frac{m^*}{2}(\dot{\vec{x}}(t))+\Delta+V(\vec{x}(t))\right]} \tag{5.27}$$

where $\vec{x}(t)$ describes the tunneling path C and $V(\vec{x})$ is the random potential. Or equivalently we may express the tunneling amplitude A in terms of the Green functions of the quasi-particle and the quasi-hole, G^p and G^h:

$$A \sim \int d^2x dt G^h\left(x, x+\frac{L}{2}; t_0, t_0+t\right) G^p\left(x, x-\frac{L}{2}; t_0, t_0+t\right) \tag{5.28}$$

If we ignore the phase factor $\oint_C d\vec{x}\cdot\vec{A}$, the second exponential in (5.27) gives rise to the factor $e^{-L\sqrt{\Delta m^*}}$ in (5.24). The summation of the phase factor $e^{i\frac{e}{q}\oint_C d\vec{x}\cdot\vec{A}}$ corresponds to the reduction factor γ in (5.24). Because the phase factor changes extremely fast from path to path, the factor γ itself may be exponentially small.

In a weak potential produced by a single impurity (*i.e.,* a potential which is non-zero only in a finite region), the quasi-particle can only do circular motion due to the strong magnetic field. The propagator of the quasi-particle is localized and takes a form $|G(x, x'; w)| \sim e^{-\#\frac{(x-x')^2}{l^2}}$ where l is the magnetic length. Therefore, we expect the total tunneling amplitude A to be a quadratic exponential in L:

$$A \sim e^{-\#\frac{L^2}{l^2}}. \tag{5.29}$$

When the potential V is periodic, the situation is very different because of possible resonance effects. Let us consider a periodic potential V such that there is a multiple of $2\pi q$ flux going through each plaquette. Such a potential changes the Landau levels of the quasi-particles into energy bands with finite width. The nontrivial dispersion relation $E(k)$ (k is the crystal momentum) implies that the quasi-particles are delocalized by the periodic potential. In other words, the wave packet of the quasi-particle moves in a straight line in presence of the periodic potential. In this case we expect the tunneling amplitude to be a linear exponential in L:

$$A \sim e^{-\#\frac{L}{l}}. \tag{5.30}$$

A more direct way to understand the above result is to notice that the easy tunneling paths favored by the periodic potential have phase factors which only differ from each other by a multiple of 2π. (Fig. 6.) There is no cancellation between the amplitudes of the easy paths. All the easy paths together contribute to the total tunneling amplitude A a term of order $e^{-\#\frac{L}{l}}$.

For generic random potentials, the quasi-particle Green function is shown to have a form[25] $|G(x, x'; w)| \sim e^{-\#\frac{|x-x'|}{\xi}}$. In this case the tunneling amplitude is expected to be given by (5.30).

The point of the above discussion is the following. The strength of the tunneling amplitude A depends on whether the quasi-particles are localized or not in the potential V. If the quasi-particles are not localized (*e.g.,* in the periodic potential), the amplitude A is expected to be of order $e^{-\#\frac{L}{l}}$. If the quasi-particles are localized (*e.g.,* in the single impurity potential), the amplitude A is expected to be smaller than $e^{-\#\frac{L}{l}}$, or more precisely, $\frac{\ln|A|}{L} \to -\infty|_{L\to\infty}$. In case of single impurity potential we further expect $A \sim e^{-\#\frac{L^2}{l^2}}$.

Before ending this section we would like to mention that Haldane[26] has suggested that the tunneling process discussed in this section may change one ground state of the FQH system to another ground state. In the topological Chern-Simons theory the algebra of the tunneling loops (or Wilson lines) (5.22) has been used to construct all the ground states.[27] Read has also pointed out that the tunneling loops satisfy the algebra (5.22) which may be used to construct the ground states.[28] These physical pictures and ideas are demonstrated explicitly in this section in the frame work of the effective G-L theory of the FQH effects.

VI. THE GROUND STATE DEGENERACY OF THE HIERARCHY FQH STATES

For general filling fraction $\nu = \frac{p}{q}$, the FQH states are given by the hierarchy scheme suggested in Ref. 29. The general hierarchy FQH states are described phenomenologically by the following effective G-L theory

$$\begin{aligned}\mathcal{L}_{GL} =&\Phi^*(i\partial_0 - a_0 - e^*A_0)\Phi + \frac{1}{2m}\Phi^*(i\partial_i - a_i - e^*A_i)^2\Phi \\ &+ V(\Phi) - \frac{\tilde{p}}{4\pi\tilde{q}}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \end{aligned} \tag{6.1}$$

with $e^* = \frac{r}{s}e$ where (r, s) and $(\tilde{p}, \tilde{q})$ are two pairs of incommensurable integers. The Hall conductance is given by $\sigma_{xy} = \frac{\tilde{p}}{\tilde{q}}\frac{e^{*2}}{\hbar} = \frac{\tilde{p}r^2}{\tilde{q}s^2}\frac{e^2}{\hbar}$ and the filling fraction $\nu = \frac{\tilde{p}r^2}{\tilde{q}s^2} = \frac{p}{q}$. We always rescale a_μ such that Φ carries unit a_μ charge. To obtain (6.1) we include the possibility that the FQH state is given by n boson condensed state $\langle\phi^n\rangle \neq 0$. Thus Φ in (6.1) may correspond to ϕ^n in (3.1). In this case $\tilde{q}$ in (6.1) can be an even integer.[28] However, an electron system may not be able to produce the general G-L theory (6.1) with all possible integer pairs (r, s) and $(\tilde{p}, \tilde{q})$. It is possible that only a subset of the integer pair (r, s) and $(\tilde{p}, \tilde{q})$ is realized by electron systems.

When $(r, s) = (1, 1)$ and $(\tilde{p}, \tilde{q}) = (1, 3)$, (6.1) describes a Laughlin state with filling factor $\nu = \frac{1}{3}$. The wave function of this FQH state is given by

$$\left[\prod_{i<j}(z_i - z_j)^3\right] e^{-\frac{1}{4}\sum\frac{|z_i|^2}{l^2}}$$

where $z_i = x_{i1} + ix_{i2}$ are coordinates of the electrons. However, when $(r, s) = (2, 1)$ and $(\tilde{p}, \tilde{q}) = (1, 12)$, (6.1) describes a different FQH state with the same fill factor $\nu = \frac{1}{3}$. Such a FQH state can be regarded as a Laughlin state for electron pairs, whose wave function is given by

$$\left[\prod_{i<j}(Z_i - Z_j)^{12}\right] e^{-\frac{1}{4}\sum\frac{2|Z_i|^2}{l^2}}$$

where Z_i are center-of-mass coordinates of the electron pairs.

We would like to remark that a more general effective G-L theory of the QH states may contain several boson fields and gauge fields, for example, it may take a form

$$\mathcal{L}_{GL} = \sum_{I=1}^{N}\left[\Phi_I^*(i\partial_0 - a_{I0} - e_I^* A_0)\Phi_I + \frac{1}{2m_I}\Phi_I^*(i\partial_i - a_{Ii} - e_I^* A_i)^2\Phi \right.$$
$$\left. +V_I(\Phi_I) - \frac{\tilde{p}_I}{4\pi\tilde{q}_I}\,\epsilon^{\mu\nu\lambda}a_{I\mu}\partial_\nu a_{I\lambda}\right]. \quad (6.1a)$$

If we choose $e_I^* = e$ and $\tilde{p}_I = \tilde{q}_I = 1$, (6.1a) describes an IQH state with N filled Landau levels. For simplicity we will concentrate on the effective G-L theory in (6.1). Most of the results obtained for (6.1) can be easily generalized so that they also apply to (6.1a).

The discussions in Sections 2 and 3 can be directly generalized to apply to (6.1). The transformations T_1 and T_2 defined in (3.7) now obey an algebra

$$T_1T_2 = e^{i\frac{2\pi\tilde{p}}{\tilde{q}}}T_2T_1. \quad (6.2)$$

The ground states of (6.1) defined on the torus have $\tilde{q}$ fold degeneracy in the thermodynamic limit.

The quasi-particle in (6.1) has a fractional statistics given by $\theta = \frac{\pi\tilde{p}}{\tilde{q}}$. The denominator of the statistical angle, is directly related to the ground state degeneracy. The statistical angle θ, however, is not directly related to the Hall conductance σ_{xy} or the filling fraction $\frac{\tilde{p}r^2}{\tilde{q}s^2} = \frac{p}{q}$.

We would like to remark that the two hierarchy FQH states given by the same $(\tilde{p}, \tilde{q})$ and different (r, s) have the same (low energy) topological structure or topological order. They only differ by a rescaling of the electric charge. On the other hand two FQH states with the same Hall conductance (and filling fraction $\frac{p}{q}$) may have different topological order corresponding to different $(\tilde{p}, \tilde{q})$ (and (r, s)).

We would like to make a side remark here. We know that in the mean field approach the anyon superfluid states[30] is closely related to the QH states. The filling fraction ν of the associated QH problem is determined by the statistical angle θ of the anyons:

$$\nu = \frac{\pi}{\theta}.$$

We know that the QH states with the same filling fraction may have different topological orders. This fact suggests that the anyon superfluid state may have different phases.[31] Each phase has different topological order. The quasi-particle excitations in different superfluid phases may have different statistics. In particular, the semion superfluid state obtained from the HQ state of two filled Landau levels does not support quasi-particles with fractional statistics, while a different semion superfluid state obtained from the (tide binding) semion pair condensation does support semionic quasi-particle excitations.[31]

We would like to emphasize that in this paper we only show that the ground states of the FQH state are at least $\tilde{q}$ fold degenerate on torus. Our proof does not exclude the possibility that the ground state degeneracy may be larger than $\tilde{q}$. However, our results do imply that the ground state degeneracy must be a multiple of $\tilde{q}$.

We would like to mention that the FQH state constructed in Ref. 29, for example, at filling fraction $\nu = \frac{2}{5}$, is described by (6.1) with $(\tilde{p}, \tilde{q}) = (1, 10)$ and $(r, s) = (2, 1)$. The quasi-particle carries a charge $\frac{e}{5}$ and has a statistics $\theta = \frac{\pi}{10}$. It is not clear whether the integer pairs $(\tilde{p}, \tilde{q}) = (2, 5)$ and $(r, s) = (1, 1)$ can be realized by (spinless) electron systems or not.

VII. DUALITY PICTURE AND APPLICATIONS TO THE CHIRAL SPIN STATES

The G-L theory (6.1) has a dual form[32,33] in which the order parameter Φ is replaced by an $U(1)$ gauge field $\tilde{a}_\mu$. Some discussions in the previous sections become more transparent in the dual theory.

To give a simple heuristic derivation of the dual theory of the G-L theory (6.1), let us first turn off a_μ and A_μ in (6.1). Now (6.1) described a superfluid state. However, the low lying excitations have a spectrum of form $\epsilon_k = \frac{k^2}{2m}$ corresponding to the free boson condensation. For interacting bosons the low lying spectrum is linear $\epsilon_k = c_s k$ which describes a phonon mode. Therefore the low lying excitation of the superfluid is rather described by

$$L = \int d^2x \frac{1}{2g^2} (\partial_\mu \chi)^2 \tag{7.1}$$

after including the interactions. In (7.1) we have set the phonon velocity $c_s = 1$. g^2 in (7.1) is the rigidness constant. χ is the phase of Φ. The superfluid current J_i is given by

$$J_i = \frac{1}{g^2} \partial_i \chi. \tag{7.2}$$

Therefore $g^2 = \frac{m}{n}$ (in the unit $c_s = 1$).

It is pointed out in Ref. 32 and 34 that (7.1) is equivalent to a $U(1)$ gauge theory described by

$$L = \int d^2x \frac{g^2}{16\pi^2} \tilde{f}_{\mu\nu}^2 \tag{7.3}$$

where $\tilde{f}_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$, if we identify the superfluid current J_i and the superfluid density $J_0 = n$ with $\epsilon_{\mu\alpha\beta} \tilde{f}^{\alpha\beta}$:

$$J_\mu = \frac{1}{4\pi} \epsilon_{\mu\alpha\beta} \tilde{f}^{\alpha\beta}. \tag{7.4}$$

A vortex in the superfluid can be viewed as a particle carrying $\tilde{a}_\mu$ charge. Including the vortex anti-vortex excitations, the effective theory may be written as

$$L = \int d^2x \left[\frac{g^2}{16\pi^2} \tilde{f}_{\mu\nu}^2 + \frac{1}{2} |(\partial_0 + i\tilde{a}_0)\Psi|^2 \right.$$
$$\left. -\frac{1}{2} c_v^2 |(\partial_i + i\tilde{a}_i)\Psi|^2 - \frac{1}{2} m_v^2 |\Psi|^2 \right] \tag{7.5}$$

The vortex density is given by $\mathrm{Re}(i\Psi^* \partial_0 \Psi)$. A Ψ particle creates an "electric" field $\tilde{f}_{i0}$ around it. From (7.4) we see that the "electric" field in radial direction corresponds to a superfluid current circulating around the Ψ particle. Thus the Ψ particle indeed generates a vortex in the superfluid. We have assigned a unit $\tilde{a}_\mu$ charge to Ψ particle such that it creates a minimum quantized vortex (see (7.4)). There is no particle carrying fractional $\tilde{a}_\mu$ charge because the circulation of a vortex is quantized. The fact that the $\tilde{a}_\mu$ charge of the excitations in the dual theory is quantized as an integer reflects that the superfluid state is a single boson Φ condensed state, *i.e.*, $\langle \Phi \rangle \neq 0$. Had the superfluid state come from N boson condensation, $\langle \Phi^N \rangle \neq 0$, the $\tilde{a}_\mu$ charge would be quantized as a multiple of $\frac{1}{N}$.

The G-L theory (6.1) of the FQH effects is obtained by coupling the superfluid current J_μ to $a_\mu + A_\mu$ and including the Chern-Simons term of a_μ. We may do the same thing to the dual theory (7.3) of the superfluid to obtain the dual theory of (6.1). After including $J_\mu(a^\mu + e^* A^\mu) - \frac{\tilde{p}}{4\pi\tilde{q}} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}$ to (7.5) we obtain the dual theory of the G-L theory

$$L_{dGL} = \int d^2x \left[\frac{g^2}{16\pi^2} \tilde{f}_{\mu\nu}^2 + \frac{1}{4\pi}(a_\mu + e^* A_\mu) \tilde{f}_{\nu\lambda} \epsilon^{\mu\nu\lambda} - \frac{\tilde{p}}{4\pi\tilde{q}} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \right.$$
$$\left. + \frac{1}{2} |(\partial_0 + i\tilde{a}_0)\Psi|^2 - \frac{1}{2} c_v^2 |(\partial_i + i\tilde{a}_i)\Psi|^2 - \frac{1}{2} m_v^2 |\Psi|^2 \right] \tag{7.6}$$

After integrating out a_μ we get

$$L_{dGL} = \int d^2x \left[\frac{g^2}{16\pi^2} \tilde{f}_{\mu\nu}^2 + \frac{e^*}{4\pi} A_\mu \tilde{f}_{\nu\lambda} \epsilon^{\mu\nu\lambda} + \frac{\tilde{q}}{4\pi\tilde{p}} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \epsilon^{\mu\nu\lambda} \right. \tag{7.7}$$
$$\left. + \frac{1}{2} |(\partial_0 + i\tilde{a}_0 \Psi|^2 - \frac{c_v^2}{2} |(\partial_i + i\tilde{a}_i)\Psi|^2 - \frac{1}{2} m_v^2 |\Psi|^2 \right]$$

Ψ describes the quasi-particle (quasi-hole) excitations above the FQH state. Due to the Chern-Simons term in (7.7), the Ψ particle (the quasi-particle) generates $\frac{2\pi\tilde{p}}{\tilde{q}}$ flux of the $\tilde{a}_\mu$ gauge field. As a bound state of charge and flux, the quasi-particle has a fractional statistics[35] $\theta = \pi\frac{\tilde{p}}{\tilde{q}}$. The quasi-particle carries fractional electric charge $\frac{\tilde{p}}{\tilde{q}} e^*$ which can be derived from the coupling $\frac{e^*}{4\pi} A_\mu \tilde{f}_{\nu\lambda} e^{\mu\nu\lambda}$ in (7.7).

To rigorously prove that (7.7) is effective theory of the FQH state, we need to prove that, after integrating out $\tilde{a}_\mu$ and Ψ, (7.7) produces the same effective Lagrangian $L_{\text{eff}}(A_\mu)$ as the electron system does. A relation similar to (3.4) should be satisfied. Although here we can not show that (3.4) is satisfied by the dual theory (7.7), the dual effective theory does reproduce (at least qualitatively) all known low energy properties of the FQH states.

Therefore we expect that the FQH states are correctly described by (7.7) at low energies and we may use (7.7) to study other (unknown) low energy properties of the FQH states.

In order to use the dual theory (7.7) to study the ground state degeneracy of the FQH states, we first need to quantize (7.7).[27] At the moment let us ignore the quasi-particle field Ψ. Following the approach in Section 4, we may quantize (7.7) in the gauge $\tilde{a}_0 = 0$. The constraint associated with the equation of motion is

$$\begin{aligned} G(x) = \frac{\delta L_{dGL}}{\delta a_0} &= \frac{g^2}{4\pi^2}\partial_i \tilde{f}^{0i} + \frac{\tilde{q}}{2\pi\tilde{p}}\epsilon^{ij}\partial_i \tilde{a}_j \\ &= \partial_i \pi^i + \frac{\tilde{q}}{4\pi\tilde{p}}\epsilon^{ij}\partial_i \tilde{a}_j = 0 \end{aligned} \tag{7.8}$$

π^i in (7.8) is the canonical momentum conjugated to a_i:

$$\pi^i = \frac{\delta L_{dGL}}{\delta \dot{a}_i} = \frac{g^2}{4\pi^2}\tilde{f}^{0i} + \frac{\tilde{q}}{4\pi\tilde{p}}\epsilon^{ij}\tilde{a}_j \tag{7.9}$$

After the quantization the operators $\hat{a}_i$ and $\hat{\pi}_i$ satisfy

$$\left[\hat{\pi}_i(x), \hat{a}_j(y)\right] = i\delta^2(x-y) \tag{7.10}$$

Under the gauge transformations $\hat{a}_i$ and $\hat{\pi}_i$ transform as

$$\begin{aligned} \hat{a}_i &\to \hat{a}_i + \partial_i f \\ \hat{\pi}_i &\to \hat{\pi}_i + \frac{\tilde{q}}{4\pi\tilde{p}}\epsilon^{ij}\partial_j f \end{aligned} \tag{7.11}$$

where f is a single valued function on the torus. Using (7.10) we see that the gauge transformation (7.11) is generated by the operator

$$\begin{aligned} G[f] &= e^{i\int d^2x \partial_i f\left(\pi^i + \frac{\tilde{q}}{4\pi\tilde{p}}\epsilon^{ij}\tilde{a}_j\right)} \\ &= e^{-i\int d^2x f G(x)} \end{aligned} \tag{7.12}$$

Once again the constraint (7.8) generate the gauge transformation. Due to the gauge invariance of the theory, all physical operators commute with $G[f]$. Noticing that $G[f]$ and $G[f']$ commute we may require the physical states (which form a representation of physical operators) to satisfy

$$G[f]|\Psi_{\text{phy}}\rangle = |\Psi_{\text{phy}}\rangle \tag{7.13}$$

The constraint is satisfied by the physical states, $G(x)|\Psi_{\text{phy}}\rangle = 0$. The condition (7.13) defines the physical Hilbert space.

The operator $G[f]$ given by (7.12) is well defined even when f is a multivalued function. In particular, $G[\alpha f_1]$ and $G[\alpha f_2]$ are well defined operators, where f_1 and f_2 are defined in (4.7) and α is a constant. Using (7.10) and (7.12) one can check that $G[\alpha f_1]$ and $G[\beta f_2]$ satisfy an algebra

$$G[\alpha f_1]G[\beta f_2] = e^{i\alpha\beta\frac{2\pi\tilde{q}}{\tilde{p}}}G[\beta f_2]G[\alpha f_1] \tag{7.14}$$

Because the $\tilde{a}_\mu$ charge is quantized as integers, this is equivalent to say that $G[f_1]$ and $G[f_2]$ generate the (large) gauge transformations and commute with all the physical operators. However, because $G[f_1]$ and $G[f_2]$ do not commute

$$G[f_1]G[f_2] = e^{i\frac{2\pi\tilde{q}}{\tilde{p}}} G[f_2]G[f_1] \tag{7.15}$$

if $\tilde{p} \neq 1$, we can not require the physical states to be invariant under both $G[f_1]$ and $G[f_2]$. But we can further restrict the physical Hilbert space by requiring the physical states to satisfy, for example,

$$\begin{aligned} G[f_1]|\Psi_{\text{phy}}\rangle &= |\Psi_{\text{phy}}\rangle \\ G^{\tilde{p}}[f_2]|\Psi_{\text{phy}}\rangle &= |\Psi_{\text{phy}}\rangle \end{aligned} \tag{7.16}$$

because $G[f_1]$ and $G^{\tilde{p}}[f_2]$ commute.

Notice that $G[\alpha f_1]$ and $G[\alpha f_2]$ commute $G[f]$. When $\alpha = \frac{\tilde{p}}{\tilde{q}}$, $G[\alpha f_1]$ and $G[\alpha f_2]$ also commute with $G[f_1]$ and $G[f_2]$. Therefore

$$\begin{aligned} T_1 &\equiv G\left[\frac{\tilde{p}}{\tilde{q}} f_1\right] \\ T_2 &\equiv G\left[\frac{\tilde{p}}{\tilde{q}} f_2\right] \end{aligned} \tag{7.17}$$

act on the physical Hilbert space defined by (7.13) and (7.16). Later we will show that T_1 and T_2 are generated by the quasi-particle tunneling described in Section 5 and they are physical operators. T_1 and T_2 satisfy the algebra

$$T_1 T_2 = e^{i\frac{2\pi\tilde{p}}{\tilde{q}}} T_2 T_1 \tag{7.18}$$

and the physical states form a representation of the algebra (7.18). The Hamiltonian of the dual theory (7.7) is given by (after ignoring Ψ field)

$$H = \int \left[\frac{g^2}{8\pi^2}(f^{0i})^2 + \frac{g^2}{8\pi^2}(f^{12})^2\right] d^2x \tag{7.19}$$

The Hamiltonian commutes with the gauge generators $G[f]$, $G[f_1]$ and $G[f_2]$. Therefore H acts on the physical Hilbert space. The Hamiltonian (7.19) also commutes with the *physical* operators T_1 and T_2. Hence the ground states of H must form a representation of the algebra (7.18) and are (at least) $\tilde{q}$ fold degenerate.

We would like to remark that the above discussions demonstrate that the topological Chern-Simons theory of compact $U(1)$ gauge field can be (mathematically) consistantly quantized, even when the coefficient in front of the Chern-Simons is a rational number. This is true at least when the space-time metrics is kept fixed.

We would like to point out that if we separate the local and the global excitations by writing

$$\tilde{a}_i = \frac{2\pi\theta_i}{L_i} + \delta\tilde{a}_i \tag{7.20}$$

from (7.4) we see that $\dot{\theta}_1$ ($\dot{\theta}_2$) corresponds to constant current density in x_2 (x_1) direction. Therefore $\epsilon^{ij}\theta_i$ are proportional to the center of mass coordinate x_{ci}. The operator T_1 and T_2 shift θ_i:

$$\theta_i \to \theta_i + 2\pi\frac{\tilde{p}}{\tilde{q}} \tag{7.21}$$

if we choose $f_i = 2\pi\frac{x_i}{L_i}$. Thus the operator T_1 (T_2) discussed in this section corresponds to the magnetic translation T_2 (T_1) discussed in Section 2 which also shifts the center of mass coordinates. However the operators T_i discussed in this section have a local definition and can be easily generalized by the high genus Riemann surface.

Now let us consider the effects of the quasi-particle fluctuations Ψ. First we notice that, for finite torus, two operators

$$\begin{aligned} W_1 &= e^{-i\int_0^{L_1} dx_1 \hat{\tilde{a}}_1} \\ W_2 &= e^{+i\int_0^{L_2} dx_2 \hat{\tilde{a}}_2} \end{aligned} \tag{7.22}$$

are invariant under the gauge transformations generated by $G[f]$, $G[f_1]$ and $G[f_2]$. There is no reason to exclude the gauge invariant term

$$\Delta H = (c_1 W_1 + c_2 W_2 + h.c.) \tag{7.23}$$

from the effective Hamiltonian. Indeed after we integrate out Ψ field (with fixed $\tilde{a}_\mu$), ΔH is induced by the quasi-particle fluctuations. It precisely comes from the quasi-particle quasi-hole tunneling process discussed in Section 5. Under T_i the operators W_i transform as

$$\begin{aligned} T_1^{-1} W_1 T_1 &= e^{-i\frac{2\pi\tilde{p}}{\tilde{q}}} W_1 \\ T_2^{-1} W_1 T_2 &= W_1 \\ T_1^{-1} W_2 T_1 &= W_2 \\ T_2^{-1} W_2 T_2 &= e^{i\frac{2\pi\tilde{p}}{\tilde{q}}} W_2 \end{aligned} \tag{7.24}$$

When restricted to the subspace spaneed by the ground states (which are assumed to form an irreducible representation of (7.18)), W_1 (W_2) can be shown to be proportional to T_2 (T_1). Because ΔH does not commute with T_i, the ground state degeneracy is lifted by the quasi-particle tunneling effects.

Using the approach in Section 5 we can show explicitly that the quasi-particle tunneling generates physical operators T_i.

On the torus the quasi-particle quasi-holes tunneling discussed in Section 5 is given by the following ansatz

$$\tilde{a}_1 + i\tilde{a}_2 = \frac{\tilde{p}}{\tilde{q}} \sum_{mn} \left[\frac{i(z - z_0 - Z_{mn})}{|z - z_0 - Z_{mn}|^2 + \xi'^2} - \frac{i(z - \tilde{z}_0 - Z_{mn})}{|z - \tilde{z}_0 - Z_{mn}|^2 + \xi'^2} \right] \tag{7.25}$$

where z_0 and $\tilde{z}_0$ satisfying (5.13) are the coordinates of the quasi-particle and the quasi-hole. After the tunneling (in x_1 direction), a initial configuration $(\tilde{a}_1, \tilde{a}_2)$ is changed to a final configuration $\left(\tilde{a}_1, \tilde{a}_2 + \frac{\tilde{p}}{\tilde{q}}\frac{2\pi}{L}\right)$. The tunneling in x_2 direction changes the configuration $(\tilde{a}_1, \tilde{a}_2)$ to $\left(\tilde{a}_1 - \frac{\tilde{p}}{\tilde{q}}\frac{2\pi}{L}, \tilde{a}_2\right)$. Let us use operators U_1 and U_2 to denote the above transformations:

$$
\begin{aligned}
U_2 &: (\tilde{a}_1, \tilde{a}_2) \to \left(\tilde{a}_1, \tilde{a}_2 + \frac{p}{q}\frac{2\pi}{L}\right) \\
U_1 &: (\tilde{a}_1, \tilde{a}_2) \to \left(\tilde{a}_1 - \frac{p}{q}\frac{2\pi}{L}, \tilde{a}_2\right)
\end{aligned}
\tag{7.26}
$$

Using the similar calculation performed in Section 5 (see (5.16)–(5.22)) we find that U_1 and U_2 satisfy

$$
U_1^{-1}U_2^{-1}U_1U_2 = e^{i\frac{2\pi\tilde{p}}{\tilde{q}}} \tag{7.27}
$$

From (7.26) and (7.27) we see that U_i are proportional to T_i.

According to Ref. 5, after setting $A_\mu = 0$, (7.7) with $\tilde{p} = 1$ and $\tilde{q}$ an even integer is precisely the effective theory of the chiral spin states. Ψ field now describes the spinon excitations. Therefore the discussions in this paper about the FQH states also apply to the chiral spin states. In particular, we find that the ground state degeneracy of the chiral spin state is very robut as suggested in Ref. 8. The degeneracy persists even when the translation symmetry is broken, *e.g.*, when the spin-spin coupling J_{ij} has a spatial dependence.

VIII. GROUND STATE DEGENERACY OF THE FQH STATES ON ARBITRARY RIEMANN SURFACE

In Ref. 7 the ground state degeneracy of the chiral spin states (described by (7.7) with $\tilde{p} = 1$) is shown to be $\tilde{q}^g$ (for a given chirality) on a Riemann surface with genus g. In this section we will derive a similar result for the FQH state. We will take the Lagrangian (6.1) as our starting point. However, on an arbitrary Riemann surface Σ_g with genus $g \neq 1$, (6.1) needs to be generalized to

$$
\begin{aligned}
\mathcal{L}_{GL} = \Phi^* i D_0 \Phi &- \frac{1}{2m} g^{ij} D_i\Phi^* D_j\Phi \\
&- V(\Phi) - \frac{\tilde{p}}{4\pi\tilde{q}}\,\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda
\end{aligned}
\tag{8.1}
$$

where $D_\mu\Phi = (\partial_\mu + ia_\mu + A_\mu)\Phi$ and $D_\mu\Phi^* = (\partial_\mu - ia_\mu - ie^* A_\mu)\Phi^*$. g^{ij} in (8.1) is a two-dimensional metrics which in general has a spatial dependence. The matrices g^{ij} is necessary because we can not choose a single coordinate patch to cover the whole Riemann surface Σ_g with $g \neq 1$. On the Riemann surface Σ_g with $g > 1$ the translation symmetry is bound to be broken.

We will use the method developed in Section 4 to derive our result. On a Riemann surface Σ_g there are $2g$ canonical 1-cycles denoted as α_a and β_a, $a = 1, \ldots, g$ (Fig. 7). We choose $2g$ functions f_b ($b = 1, \ldots, 2g$) on Σ_g such that f_a has a 2π jump along α_a and f_{g+a}

has a 2π jump along β_a, where $a = 1, \ldots, g$. However, we require $\partial_i f_a$ to be a smooth vector field on Σ_g. Using f_a we define unitary operators T_a as the following

$$\begin{aligned} T_a &= e^{i \int d^2x f_a(x) \hat{G}(x)} \\ &= e^{-i \int d^2x \hat{\Phi}^\dagger \hat{\Phi} f_a} e^{-i \frac{\tilde{p}}{2\pi\tilde{q}} \int d^2x a_i \partial_j f_a \epsilon^{ij}} \\ a &= 1, \ldots, 2g \end{aligned} \tag{8.2}$$

After a transformation by T_a, $\hat{\Phi} \to \hat{\Phi}' = e^{if_a}\hat{\Phi}$. $\hat{\Phi}'$ remain to be a smooth function on Σ_g. Using the commutation relation

$$[\hat{a}_1(x), \hat{a}_2(y)] = i2\pi \frac{\tilde{q}}{\tilde{p}} \delta^2(x - y) \tag{8.3}$$

we find that

$$T_a T_b = e^{i \frac{p}{2\pi q} \int d^2x \partial_i f_a \partial_j f_b \epsilon^{ij}} T_b T_a \tag{8.4}$$

The exponent in (8.4) can be evaluated and we find

$$\begin{aligned} \int d^2x \partial_i f_a \partial_j f_b \epsilon^{ij} &= (2\pi)^2 \eta_{ab} \\ (\eta_{ab}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bigotimes I_{g\times g} \end{aligned} \tag{8.5}$$

where $I_{g\times g}$ is a $g \times g$ unit matrix. Therefore (8.4) can be rewritten as

$$\begin{aligned} T_a T_{g+a} &= e^{i \frac{2\pi\tilde{p}}{\tilde{q}}} T_{g+a} T_a \quad , \quad a = 1, \ldots, g \\ [T_a, T_b] &= 0 \quad , \quad b \neq a + g \quad , \quad a, b = 1, \ldots, 2g \end{aligned} \tag{8.6}$$

where we have assumed $T_{a+2g} = T_a$. The pairs of operators T_a and T_{g+a} generate g copies of the algebra (6.2), which commute with each other. Each copy of the algebra contributes a factor $\tilde{q}$ to the ground state degeneracy. The total ground state degeneracy is $\tilde{q}^g$.

If we compactify the space into g copies disconnected tori, the ground states of (6.1) are obviously $\tilde{q}^g$ fold degenerate. The result in this section implies that the ground state degeneracy is unchanged after we connect the g tori by tubes to form a genus g Riemann surface (Fig. 8).

IX. DISCUSSIONS

In this paper we show that the FQH states on the Riemann surface Σ_g have $\tilde{q}^g$ fold degenerate ground states if the quasi-particles in the FQH states have fractional statistics $\theta = \frac{\pi\tilde{p}}{\tilde{q}}$. The fact that the ground state degeneracy depends on the topology of the space suggests that the degeneracy is not due to the broken symmetry. We also show that the ground state degeneracy (in the thermodynamic limit) is robust against arbitrary perturbations. This means that the ground state degeneracy remains a constant in a finite region in the phase space. Therefore we may use the ground state degeneracy to characterize different phases in the phase space. We may say that the phases with different ground state degeneracy have different topological orders. As we change the coupling constants in the theory, the ground state degeneracy may jump which signals a phase transition between two phases with different topological orders.

If one insists on a symmetry breaking picture, one may regard the ground state degeneracy considered in this paper as a result of broken "topological" symmetries. The topological symmetry transformation is defined as the following. Consider a FQH state on a torus. We adiabatically add a unit flux through the hole of the torus (Fig. 9a). The Hamiltonian is invariant after adding a unit flux. Therefore the adiabatic process changes one ground state of the FQH state to another. Such a transformation can be represented by a unitary operator U_1 which acts on the ground states. Similarly, the adiabatic turning on a unit flux going through the tube of the torus (Fig. 9b) generates an operator U_2 acting on the ground states. We call the operators U_1 and U_2 the topological symmetry transformations. Notice that the topological symmetry transformations can be defined only after we specify the topology of the space. The very existence of the topological symmetry depend on the topology of the space. On the sphere there is no topological symmetry. That is why the ground state of the FQH states is non-degenerate on the sphere. On the Riemann surface Σ_g of genus g, there are $2g$ topological symmetry transformations. From Ref. 13 we find that the operators U_1 and U_2 satisfy the algebra

$$U_1^{-1}U_2^{-1}U_1U_2 = e^{i\frac{2\pi p}{q}} \tag{9.1}$$

where $\frac{p}{q}$ is the filling fraction. Therefore U_1 and U_2 can not be the identity in the subspace spanned by the ground states. This implies that the topological symmetry is spontaneous broken.

On a finite system, the ground state degeneracy may be lifted by finite size effects. For the degenerate ground states associated with ordinary symmetry breaking, the energy split is expected to be of order $e^{-\frac{L^2}{\xi^2}}$ where ξ is a microscopic length scale of the theory and L is the size of system. This is because the different ground states associated with the broken symmetry can only be connected by a tunneling process in which a domain wall sweeps over the whole system (Fig. 10). Such a domain-wall-tunneling process has an amplitude of order $e^{-\frac{L^2}{\xi^2}}$. However, the different ground states associated with the broken *topological* symmetry can be connected by the particle tunneling process (see Section 5). In this case the energy split is given by $e^{-\frac{L}{\xi}}$. Such an energy split also indicates that the ground state degeneracy of the FQH state is not due to the ordinary broken symmetry.

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FIGURE CAPTIONS

Figure 1: Some of the Feynman diagrams which contribute to the second and the third terms in (3.9).

Figure 2: $f_1(x)$ and $f_2(x)$ have a 2π jump along the two loops 1 and 2 respectively.

Figure 3: A solinoid (represented by the dotted lines) creates a quasi-particle and a quasi-hole when pearing through the torus. The particle-hole tunnel process discussed here can be viewed as a solinoid cutting through the torus. Such a process adds unit flux to the hole, which changes the winding number of ϕ going around the hole.

Figure 4: The four particle-hole tunnel processes are represented by the four directed paths in the space-time. ABCD represents the torus. AB is identified with CD and BC is identified with AD.

Figure 5: The four tunnel path in Fig. 4 can be deformed into two linked loops.

Figure 6: The dotted lines represent the minima of the potential. The two easy paths (a) and (b) favored by the potential enclose an integer number of plaquettes. The phase of tunneling amplitudes of (a) and (b) only differ by a multiple of 2π.

Figure 7: A Riemann surface and its canonical 1-cycles α_a and β_a (for $g = 3$).

Figure 8: A genus 2 Riemann surface is formed by connecting two tori by a tube.

Figure 9: A torus with flux going (a) through the hole and (b) through the tube.

Figure 10: Two ground states resulting from a broken symmetry can be connected by a domain wall tunneling process in which a domain sweeps over the whole system.