

Fermions, strings, and gauge fields in lattice spin models

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In this paper, we investigate the general properties of lattice spin models with emerging fermionic excitations in any dimensions. We argue that fermions always come in pairs and their creation operator always has a string-like structure with the newly created particles appearing at the end-points of the string. The physical implication of this structure is that the fermions always couple to a nontrivial gauge field. We present exactly soluble examples of this phenomenon in 2 and 3 dimensions. Our analysis is based on a novel algebraic formula which allows one to compute the statistics of a lattice particle in terms of its hopping operators. This algebra allows us to calculate the statistics of a particle from the properties of the corresponding string operator.

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I. INTRODUCTION

For many years, it was thought that Fermi statistics were fundamental, in the sense that one could only obtain a theory with fermionic excitations by introducing them by hand (via anti-commuting fields). Then, over the past two decades, this view began to change. A number of real world and theoretical examples showed that fermions and anyons could *emerge* as low energy collective modes of purely bosonic systems. The first examples along these lines were the fractional quantum Hall states. [1, 2] Usually we think of the FQH states as examples of anyonic excitations emerging from interacting fermions. However, from a purely theoretical point of view, the same effects should occur in systems of interacting bosons in a magnetic field.[3] There were also indications of emerging fermions in the slave-boson approach to spin-1/2 systems[4–10] and in the study of resonating valence bond (RVB) states.[11–14] Unfortunately, the RVB picture and the slave boson approach both rely on approximate or mean-field techniques to construct and analyze these exotic states. More recently, a number of researchers have introduced exactly soluble or quasi-exactly soluble models with emerging fermions. [14–21] These models allow for a more well-controlled analysis, albeit in very specific cases.

The mean-field approach and the exactly-soluble examples both provide clues to the structure and basic properties of bosonic models with fermionic excitations. They indicate, among other things, that fermions never appear alone in lattice spin systems. Instead they always come together with a nontrivial gauge field and the emerging fermions are associated with the deconfined phase of the gauge field.[7, 9, 10] The deconfined phases always seem to contain a new kind of order - topological order,[10, 22] and the emerging fermions and anyons are intimately related to the new order.

This was particularly apparent in the context of the slave-boson approach. In this technique, one expresses a spin-1/2 Hamiltonian in terms of fermion fields and

gauge fields.[4, 5] Clearly, the presence of fermion fields does not, by itself, imply the existence of fermionic quasi-particles - the fermions appear only when the gauge field is in the *deconfined* phase. Ref. [6–10] constructed several deconfined phases where the fermion fields do describe well defined quasi-particles. Depending on the properties of the deconfined phases, these quasi-particles can carry fractional statistics (for the chiral spin states) [6–8] or Fermi statistics (for the Z_2 deconfined states).[9, 10]

Although it was less evident initially, a similar picture has emerged from the study of RVB states and the associated quantum dimer models. Ref. [11, 12] originally proposed that fermions (spinons) could emerge from a nearest neighbor dimer model on a square lattice. Later, it was realized that the fermions coupled to a Z_2 gauge field, and the fermionic excitations only appear when the field is in the deconfined phase.[13] It turns out that the dimer liquid on square lattice with only nearest neighbor dimers is not in the deconfined phase (except at a critical point).[12, 13] However, on a triangular lattice, the dimer liquid does have a Z_2 deconfined phase.[14] The results in Ref. [11, 12] are valid in this case and fermionic quasi-particles do emerge in a dimer liquid on a triangular lattice.

In this paper, we attempt to clarify these observations and to put them on firmer foundations. We give a general argument which shows that emerging fermions always occur together with a nontrivial (deconfined) gauge field. In addition, we derive a novel algebraic formula which allows one to calculate the statistics of a lattice particle in terms of its hopping operators. We feel that this formula both elucidates the fundamental meaning of statistics in a lattice system and simplifies their computation. This approach has the added advantage that it works in any number of dimensions - unlike the flux binding picture[3] developed in FQH theory. We would like to point out that all the previous examples of emerging fermions are two dimensional models. The emerging fermions in those models are related to the flux binding picture in Ref. [3]. The new approach allows us to establish, for the first time, the emergence of fermions in a 3D bosonic model.

Our paper is organized as follows: in the first section, we give a definition of the statistics of a lattice particle. In the second section, we derive the algebraic statistics

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formula discussed above. In the third section, we apply the formula to the case of fermionic (or anyonic) excitations in a lattice spin system. The formula demands that the fermions always come in pairs, and that their pair creation operator has a string-like structure with the newly created particles appearing at the ends. We show that the strings represent gauge fluctuations, and the fermions carry the corresponding gauge charge. After presenting the general argument, we devote the last two sections to exactly soluble examples of this phenomenon in 2 and 3 dimensions.

The above result suggests a very interesting picture of emerging fermions. Fermions and gauge fields appear to be two sides of the same coin: fermions are the ends of strings and gauge fields are the fluctuations of strings. Extended string-like structures seem to be the key to understanding both Fermi statistics and gauge fields.

II. DEFINITION OF STATISTICS

What do we mean when we say that a particle is a “boson” or a “fermion”? In this paper, we will use a definition based on the path integral formulation of quantum mechanics. The idea is that any action S which describes an m particle system with short-range interactions, can be divided into two pieces: a local term S_{loc} which includes the usual kinetic and potential terms and depends on the local properties of the path, and a topological term, S_{top} , which only depends on the homotopy class of the path. In 3 or more dimensions, the form of S_{top} is highly constrained. Specifically, the amplitude for a closed path is always of the form $e^{iS_{top}} = (\pm 1)^n$ where n is the number of particle exchanges that occur over the course of the path. If this sign is a $+$, we say the particles are “bosons”; if the sign is a $-$, the particles are “fermions”. In 2 dimensions, there are other, more complicated possibilities for S_{top} . These particles are called “anyons.”

According to this definition, the statistics are completely determined by the topological term S_{top} in the action. One way to isolate this term is to compare the amplitude for two paths which are the same locally (this will be made more precise in the next section) but differ in their global properties. In particular, we can compare the amplitude for a path which exchanges two particles with another path which is the same locally, but doesn’t exchange the two particles. The difference in phase is then precisely $e^{iS_{top}} = (\pm 1)$, the statistical phase.

We now reformulate this (theoretical) test of statistics so that it can be applied to particles on a lattice. The prescription is as follows: take a two particle state $|r_1, s_1\rangle$, and consider a product of hopping amplitudes along a lattice path which exchanges the particles

$$\langle r_n, s_n | H | r_{n-1}, s_{n-1} \rangle \langle r_{n-1}, s_{n-1} | H | r_{n-2}, s_{n-2} \rangle \dots \langle r_3, s_3 | H | r_2, s_2 \rangle \langle r_2, s_2 | H | r_1, s_1 \rangle \quad (1)$$

Each hopping amplitude in this expression involves one of the two particles moving to a neighboring site while

the other particle remains fixed. At the end, the two particles have exchanged places: $r_n = s_1, s_n = r_1$.

Now construct another path which is the same locally, but doesn’t exchange the two particles, and take a product of hopping amplitudes along this path. The claim is that the difference between the phases of these two expressions is precisely the statistical phase of the particles. That is, the particles are fermions, bosons, or anyons depending on whether the difference in phase is $+1$, -1 , or something else.

One way to see this is to remember the derivation of the path integral formulation of quantum mechanics. According to the standard derivation, the amplitude for a two particle path $|r(t), s(t)\rangle$ is given by a product

$$\langle r(t_n), s(t_n) | e^{-iH\Delta t} | r(t_{n-1}), s(t_{n-1}) \rangle \dots \langle r(t_3), s(t_3) | e^{-iH\Delta t} | r(t_2), s(t_2) \rangle \langle r(t_2), s(t_2) | e^{-iH\Delta t} | r(t_1), s(t_1) \rangle$$

in the limit that $\Delta t = t_n - t_{n-1} \rightarrow 0$.

Now, in the discrete (lattice) case, the above expression can be further simplified. Since $\Delta t \rightarrow 0$, we can rewrite it as

$$\langle r(t_n), s(t_n) | (1 - iH\Delta t) | r(t_{n-1}), s(t_{n-1}) \rangle \dots \langle r(t_3), s(t_3) | (1 - iH\Delta t) | r(t_2), s(t_2) \rangle \langle r(t_2), s(t_2) | (1 - iH\Delta t) | r(t_1), s(t_1) \rangle$$

Successive states $|r(t_i), s(t_i)\rangle, |r(t_{i+1}), s(t_{i+1})\rangle$ must either be identical, or must differ by a single particle hop. The matrix elements between identical states don’t contribute to the phase. Thus, we can drop them without affecting our result. We are left with:

$$\langle r(t'_k), s(t'_k) | (-iH\Delta t) | r(t'_{k-1}), s(t'_{k-1}) \rangle \dots \langle r(t'_3), s(t'_3) | (-iH\Delta t) | r(t'_2), s(t'_2) \rangle \langle r(t'_2), s(t'_2) | (-iH\Delta t) | r(t'_1), s(t'_1) \rangle$$

where the t'_k s are all distinct.

Now, as we discussed earlier, the statistical phase can be obtained by comparing the phase of this product with another product, which is the same locally, but doesn’t exchange the two particles. When we make this comparison, the phase factors of $-i$ drop out (since they contribute equally to the two products). Thus, we just need to compare products of the form

$$\langle r(t'_k), s(t'_k) | H | r(t'_{k-1}), s(t'_{k-1}) \rangle \dots \langle r(t'_3), s(t'_3) | H | r(t'_2), s(t'_2) \rangle \langle r(t'_2), s(t'_2) | H | r(t'_1), s(t'_1) \rangle$$

to determine the statistics of the particles.

III. STATISTICS AND THE HOPPING OPERATOR ALGEBRA

In this section we derive a simple algebraic formula for the statistics of a particle hopping on a lattice. This formula is completely general and holds irrespective of

whether the particles are fundamental or are simply low energy excitations of an underlying condensed matter system (e.g. quasi-particles).

We begin with a Hilbert space which describes n hard-core particles hopping on a d dimensional lattice. The states can be labeled by listing the positions of the n particles: $|i_1, i_2, \dots, i_n\rangle$. The particles are identical so the states $|i_1, i_2, \dots, i_n\rangle$ do not depend on the order of i_1, i_2, \dots, i_n . For example,

$$|i_1, i_2, \dots, i_n\rangle = |i_2, i_1, \dots, i_n\rangle$$

A typical Hamiltonian for this system is of the form

$$H = \sum_{\langle ij \rangle} (t_{ij} + t_{ji})$$

where t_{ij} are ‘‘hopping operators’’ with the property that

$$t_{ij}|j, i_1, \dots, i_{n-1}\rangle \propto |i, i_1, \dots, i_{n-1}\rangle$$

We assume that the hopping is local, *i.e.*

$$[t_{ij}, t_{kl}] = 0$$

if i, j, k, l are all different.

Our goal is to compute the statistics of the particles described by this hopping Hamiltonian H . In the following, we show that the statistical angle can be derived from the simple algebraic properties of the hopping operators. More precisely, we show that the particles obey statistics $e^{i\theta}$ if

$$t_{il}t_{ki}t_{ij} = e^{i\theta}t_{ij}t_{ki}t_{il} \quad (2)$$

for any three hopping operators t_{ij}, t_{ki}, t_{il} , where j, k, l are (distinct) neighbors of i (ordered in the clockwise direction if in 2 dimension). The orientation convention in 2 dimensions is necessary for the anyonic case.

The simplest examples where we can apply this formula are the cases of noninteracting hard-core bosons or fermions. In these cases the n particle Hamiltonian can be written as

$$H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + c_j^\dagger c_i)$$

where the c_i 's are the boson or fermion annihilation operators. The hopping operators are just $t_{ij} = -tc_i^\dagger c_j$. A little algebra confirms that the boson and fermion Hamiltonians satisfy Eq. (2) with $e^{i\theta} = +1$ and -1 , respectively.

We now give a general derivation of the formula. We begin with the state $|i, j, \dots\rangle$ which contains particles at sites i, j and other particles far away. Imagine that the two particles at i, j swap positions via an appropriate product of hopping operators.

The swapped state will be of the form

$$|i, j, \text{swapped}\rangle = [t_{jj'}t_{j'p}\dots t_{qr}]\{t_{i\alpha}t_{\alpha\beta}\dots t_{\gamma j'}t_{j'j}\}[t_{rs}\dots t_{ti}]|i, j, \dots\rangle$$

Here, we've put the hopping operators involving particle i in square brackets and the hopping operators involving

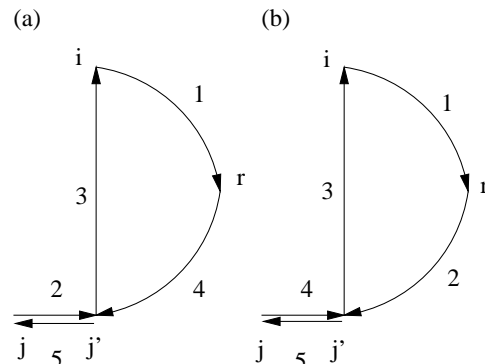


FIG. 1: (a) The path of the particles in the swapped state $|i, j, \text{swapped}\rangle$. The numbers label the order in which the paths are traversed. (b) The path of the particles in the unswapped state $|i, j, \text{unswapped}\rangle$. Notice that the two states only differ in the ordering of the paths. However, in one case the particles switch places, while in the other, they do not.

particle j in curly brackets to make this expression easier to understand (see Fig. 1 a).

Now, compare this swapped state to the following unswapped state:

$$|i, j, \text{unswapped}\rangle = \{t_{jj'}t_{j'j}\}[t_{i\alpha}t_{\alpha\beta}\dots t_{\gamma j'}t_{j'p}\dots t_{qr}t_{rs}\dots t_{ti}]|i, j, \dots\rangle$$

In this state, the two particles simply move along independent loops, without any swapping taking place (see Fig. 1 b).

We're interested in comparing the phases of these two states. Before we do this, notice that the two states involve the same product of hopping operators. The only difference is the order of these operators. This means that, in the swapped and unswapped states, the two particles trace out the same total path in (internal and external) configuration space. This is important because it means that any phase which comes from gauge fields or other Berry phases contributes equally to the swapped and unswapped states. *The difference between the two phases is therefore exactly equal to the statistical phase $e^{i\theta_{\text{stat}}}$, which we wish to compute.*

This intuitive argument can be made rigorous. Indeed, it's not hard to see that the two space-time paths traced out by the swapped and unswapped states are exactly the same locally - they only differ by a global rearrangement in space and time. Furthermore, the relative phase of the swapped and unswapped states is precisely the difference between the phases of expressions of the form (1). Thus, our discussion in the previous section *implies* that the phase difference is exactly the statistical phase.

To compute this relative phase, we use the assumed algebraic relation

$$t_{j'p}t_{\gamma j'}t_{j'j} = e^{i\theta}t_{j'j}t_{\gamma j'}t_{j'p}$$

Applying this relation to the swapped state and reordering the hopping operators using locality, we find

$$|i, j, \text{swapped}\rangle = e^{i\theta}|i, j, \text{unswapped}\rangle$$

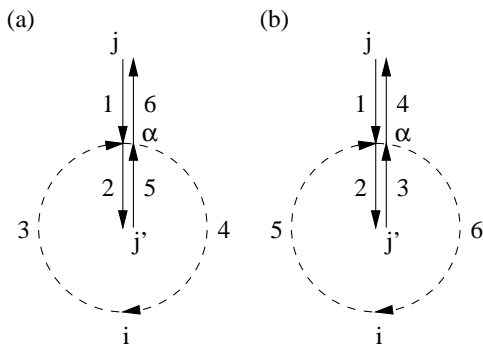


FIG. 2: (a) The path of the particles in the wound state $|i; j, \text{wound}\rangle$. The dotted line is the path of particle 1, and the solid line is the path of particle 2. The numbers label the order in which the paths are traversed. (b) The path of the particles in the unwound state $|i; j, \text{unwound}\rangle$. Notice that the two states only differ in the ordering of the paths. However, in one case the particles wind around each other, while in the other, they do not.

We see that the difference between the two phases is $e^{i\theta}$, so that $e^{i\theta_{\text{stat.}}} = e^{i\theta}$ as claimed.

Next, we consider the problem of relative statistics. We begin with a Hilbert space which describes two types of hard-core particles hopping on a 2D lattice. For concreteness, say that there are m particles of type 1 and n particles of type 2. As before, the states can be labeled by listing the positions of the two types of particles: $|i_1, \dots, i_m; j_1, \dots, j_n\rangle$.

A typical Hamiltonian for this system, is of the form

$$H = \sum_{\langle ij \rangle} (t_{ij}^1 + t_{ji}^1 + t_{ij}^2 + t_{ji}^2)$$

where t_{ij}^1, t_{ij}^2 are hopping operators for the two types of particles.

We wish to calculate the relative statistics of the two types of particles. As before, the statistics are related to the simple algebraic properties of the hopping operators. Specifically, we will show that the particles have relative statistics $e^{i\phi}$ if

$$(t_{ip}^2 t_{pj}^2)(t_{kp}^1 t_{pl}^1) = e^{i\phi} (t_{kp}^1 t_{pl}^1)(t_{ip}^2 t_{pj}^2) \quad (3)$$

Here, i, k, j, l are (distinct) neighbors of p oriented in the clockwise direction.

We begin with the state $|i; j, \dots\rangle$ with a type 1 particle at i , a type 2 particle at j , and other particles which are far away.

Consider what happens when particle 1 winds around particle 2 via an appropriate product of hopping operators (see Fig. 2a):

$$|i; j, \text{wound}\rangle = t_{j\alpha}^2 t_{\alpha j'}^2 (t_{i\ell}^1 \dots t_{m\alpha}^1 t_{\alpha n}^1 \dots t_{pq}^1 t_{qi}^1) t_{j'\alpha}^2 t_{\alpha j}^2 |i; j, \dots\rangle$$

Compare this state to the following state where the two particles don't wind around each other (see Fig. 2b):

$$|i; j, \text{unwound}\rangle = (t_{i\ell}^1 \dots t_{m\alpha}^1 t_{\alpha n}^1 \dots t_{pq}^1 t_{qi}^1) t_{j\alpha}^2 t_{\alpha j'}^2 t_{j'\alpha}^2 t_{\alpha j}^2 |i; j, \dots\rangle$$

As before, the two states involve the same path in internal and external configuration space. Thus, the phase

difference of the states is precisely the relative statistics $e^{i\phi_{\text{rel.}}}$. This can be made rigorous using an argument similar to the exchange statistics case.

We calculate this difference, using the assumed algebraic relation

$$(t_{j\alpha}^2 t_{\alpha j'}^2)(t_{m\alpha}^1 t_{\alpha n}^1) = e^{i\phi} (t_{m\alpha}^1 t_{\alpha n}^1)(t_{j\alpha}^2 t_{\alpha j'}^2)$$

Applying this relation, and reordering the hopping operators using locality, we find

$$|i; j, \text{wound}\rangle = e^{i\phi} |i; j, \text{unwound}\rangle$$

This establishes the desired result, $e^{i\phi_{\text{rel.}}} = e^{i\phi}$.

A good consistency check for these formulas can be obtained by considering the relative statistics of *two particles of the same type*. In this case, we expect the angle for the relative statistics to be exactly twice the angle for the exchange statistics:

$$e^{i\phi_{\text{rel.}}} = e^{2i\theta_{\text{stat.}}}$$

One way to see this is that exchanging the particles twice in the same direction is topologically equivalent to winding one particle around the other.

This result, which has a topological character to it, can be derived algebraically from our two formulas. We start with the expression $(t_{ip} t_{pj})(t_{kp} t_{pl})$. Applying the exchange statistics formula Eq. (2), we find:

$$\begin{aligned} (t_{ip} t_{pj})(t_{kp} t_{pl}) &= t_{ip} (t_{pj} t_{kp} t_{pl}) \\ &= t_{ip} (e^{i\theta_{\text{stat.}}} t_{pl} t_{kp} t_{pj}) \end{aligned}$$

Applying the formula again gives

$$\begin{aligned} t_{ip} (t_{pl} t_{kp} t_{pj}) &= (t_{ip} t_{pl} t_{kp}) t_{pj} \\ &= (e^{i\theta_{\text{stat.}}} t_{kp} t_{pl} t_{ip}) t_{pj} \end{aligned}$$

Combining these two equations gives

$$(t_{ip} t_{pj})(t_{kp} t_{pl}) = e^{2i\theta_{\text{stat.}}} (t_{kp} t_{pl})(t_{ip} t_{pj})$$

Comparing this with the relative statistics formula Eq. (3), we see that $e^{i\phi_{\text{rel.}}} = e^{2i\theta_{\text{stat.}}}$, as claimed.

IV. FERMIONS AND STRINGS

In this section we consider the properties of a bosonic system with fermionic or anyonic excitations. We argue that the excitations are always created in pairs, and the creation operator for a pair of particles has a string-like structure, with the new particles located at the ends. One interpretation of this is that fermions never appear alone - they always come with some kind of gauge field.

Suppose we have a bosonic model with fermionic excitations (the anyonic case is completely analogous). We expect that the Hilbert space contains a low energy subspace which is spanned by n particle states (this low energy subspace corresponds to the full Hilbert space in the previous section). The Hamiltonian for the bosonic model is typically of the form

$$H = \sum_{\langle ij \rangle} (t_{ij} + t_{ji}) + \dots$$

where the t_{ij} act like hopping operators when restricted to the n particle subspace. Here, we've left out the terms which act trivially within the low energy subspace.

We expect that the t_{ij} are local in the underlying bosonic degrees of freedom. That is, t_{ij} is a product of two operators acting within the local Hilbert spaces at sites i and j . This implies that t_{ij} and t_{kl} commute when $i, j \neq k, l$. Using the argument in the previous section, we conclude that the t_{ij} satisfy the relation

$$t_{jl}t_{ij}t_{jk} = -t_{jk}t_{ij}t_{jl} \quad (4)$$

when restricted to the low energy subspace.

We will now show that this algebraic relation demands that the fermionic creation operator has a string-like structure with the newly created particles located at the ends. More specifically, consider the following product of hopping operators:

$$W = t_{ij}t_{jk}t_{kl}\dots t_{pq}t_{qr}$$

This operator creates a fermion at i and destroys a fermion at r . Our claim is that this string operator cannot be trivial - that is, it must act non-trivially on the degrees of freedom along the path joining i and r .

To see why this is so, suppose otherwise. Consider the (very short) string $t_{ij}t_{jl}$. By our assumption, this string only acts at the sites i, l . It has no effect at j . Moreover, by locality, we expect $t_{ij} = A_i B_j$ for some operators A, B acting within the local Hilbert space at sites i, j . The only way that $t_{ij}t_{jl}$ can have no effect at j is if $A \propto B^{-1}$. Thus, $t_{ij} \propto A_i A_j^{-1}$. Substituting this into the swapping statistics formula, we find

$$\begin{aligned} t_{jl}t_{ij}t_{jk} &= A_j A_l^{-1} A_i A_j^{-1} A_j A_k^{-1} \\ &= A_j A_k^{-1} A_i A_j^{-1} A_j A_l^{-1} \\ &= t_{jk}t_{ij}t_{jl} \end{aligned}$$

This contradicts the fact that the particles are fermions (Eq. (4)). Our assumption must be false, and we conclude that W acts non-trivially on the degrees of freedom along the string.

This argument is a bit sketchy, but the basic idea is clear - if the string operator is trivial and only acts on the degrees of freedom near the ends, then the swapping statistics formula demands that the particles are bosons.

The presence of this nontrivial string indicates that fermions always appear together with some kind of gauge field. One way to see this is to consider a closed loop of hopping operators $t_{ij}t_{jk}\dots t_{pq}t_{qi}$. This string can be interpreted as a Wilson loop operator, since its phase is precisely the accumulated phase of the particle when it traverses a loop. The fact that it is nontrivial (that is, not equal to the identity operator) means that the particle is coupled to a nontrivial gauge field.

V. A 2D EXAMPLE

In this section, we present an exactly soluble lattice spin model with fermionic excitations. The exactly soluble model provides a concrete realization of the string

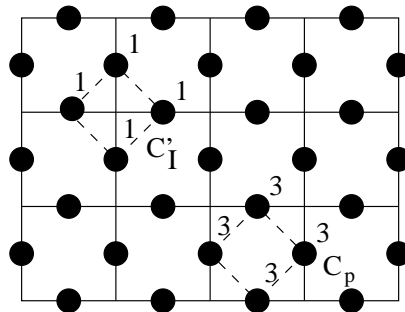


FIG. 3: A schematic diagram depicting Kitaev's model. Each spin is drawn as a dot. The paths C'_I and C_p are drawn as dotted diamonds, and are labeled with 1's or 3's according to whether the corresponding term in the Hamiltonian involves σ^1 's or σ^3 's.

picture discussed above. The emerging fermions turn out to be coupled to a Z_2 gauge field.

In this spin-1/2 system, first proposed by Kitaev[15], the spins live on the *links* of a square lattice (Fig. 3). The Hamiltonian is

$$H = -U \sum_I \left(\prod_{C'_I} \sigma_j^1 \right) - g \sum_p \left(\prod_{C_p} \sigma_j^3 \right) \quad (5)$$

Here i labels the links, I labels the sites, and p labels the plaquettes of the square lattice. Also, C'_I denotes the loop connecting the four spins adjacent to site I , while C_p denotes the loop connecting the four spins adjacent to plaquette p (see Fig. 3).

This model is exactly soluble since all the terms in the Hamiltonian commute with each other. The ground state satisfies

$$\prod_{C'_I} \sigma_j^1 = 1$$

for all sites I , and

$$\prod_{C_p} \sigma_j^3 = 1$$

for all plaquettes p . There are two types of (localized) excited states. We can have a site where

$$\prod_{C'_I} \sigma_j^1 = -1$$

or we can have a plaquette where

$$\prod_{C_p} \sigma_j^3 = -1$$

We call the first type of excitation a ‘‘charge’’ and the second type of excitation a ‘‘flux.’’ Static charge and flux configurations are exact eigenstates of the above Hamiltonian. Thus, the charge and flux quasi-particles have no dynamics.

This lack of dynamics is a special feature of the above model. However, we are interested in the properties of a *generic* Hamiltonian in the same quantum phase. Thus,

we need to perturb the system, and analyze the resulting dynamics. The simplest nontrivial perturbation is

$$H' = H + J_1 \sum_i \sigma_i^1 + J_3 \sum_i \sigma_i^3 \quad (6)$$

It's not hard to see that the first term allows the fluxes to hop from plaquettes to adjacent plaquettes, while the second term allows the charges to hop from sites to neighboring sites.

First, we calculate the statistics of the fluxes. To do this, we restrict our Hamiltonian to the low energy subspace with n fluxes and zero charges. Within this subspace, our Hamiltonian reduces to

$$H'_{\text{eff}} = J_1 \sum_i \sigma_i^1$$

To make contact with our previous formalism, we write this as

$$H'_{\text{eff}} = \sum_{\langle \mathbf{p}\mathbf{q} \rangle} (t_{\mathbf{p}\mathbf{q}} + t_{\mathbf{q}\mathbf{p}})$$

where the sum is taken over adjacent plaquettes, \mathbf{p}, \mathbf{q} , $t_{\mathbf{p}\mathbf{q}}$ is defined by $t_{\mathbf{p}\mathbf{q}} = \frac{J_1}{2} \sigma_i^1$, and \mathbf{i} the link joining \mathbf{p} and \mathbf{q} .

To calculate the statistics, we need to compare $t_{\mathbf{p}\mathbf{q}} t_{\mathbf{r}\mathbf{p}} t_{\mathbf{p}\mathbf{s}}$ with $t_{\mathbf{p}\mathbf{s}} t_{\mathbf{r}\mathbf{p}} t_{\mathbf{p}\mathbf{q}}$. Well, it's obvious from the definition that all the hopping operators $t_{\mathbf{p}\mathbf{q}}$ commute with one another. Therefore,

$$t_{\mathbf{p}\mathbf{q}} t_{\mathbf{r}\mathbf{p}} t_{\mathbf{p}\mathbf{s}} = t_{\mathbf{p}\mathbf{s}} t_{\mathbf{r}\mathbf{p}} t_{\mathbf{p}\mathbf{q}}$$

so $e^{i\theta_{\text{stat.}}} = 1$. We conclude that the fluxes are bosons. In the same way, one can show that the charges are also bosons.

Next, we consider the bound state of a flux and a charge. That is, we consider excitations with a flux through some plaquette \mathbf{p} and also a charge at one of the sites \mathbf{I} adjacent to \mathbf{p} .

These bound states are not actually stable for the above Hamiltonian (6) - the charge and flux will separate from one another over time. However, one can imagine modifying the Hamiltonian so that charges and fluxes prefer to be adjacent to each other. In this case, the bound state is a true quasi-particle.

Suppose we've made such a modification. We can then consider the statistics of the bound state. If we restrict our Hamiltonian (6) to the low energy subspace with n bound states, then our Hamiltonian reduces to

$$H'_{\text{eff}} = J_1 \sum_i \sigma_i^1 + J_3 \sum_i \sigma_i^3$$

To understand the effect of these terms, imagine we have a bound state with a flux at \mathbf{p} and a charge at \mathbf{I} . It's not hard to see that the first term allows the flux to hop to the two neighboring plaquettes which are also adjacent to \mathbf{I} . Similarly, the second term allows the charge to hop to the two neighboring sites, adjacent to \mathbf{p} . All other hopping destroys the bound state and is therefore forbidden for energetic reasons. Thus, the perturbation gives rise to

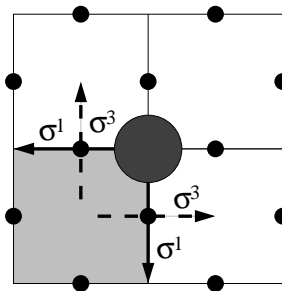


FIG. 4: A bound state of a charge, denoted by a dark circle, and a flux, denoted by a shaded square. The arrows illustrate the four ways the bound state can hop. The two charge hops are denoted by solid arrows, and have hopping operator σ^3 . The flux hops are denoted by dotted arrows, and have hopping operator σ^1 . Notice that each charge hopping operator anti-commutes with a corresponding flux hopping operator, but everything else commutes.

4 types of hopping operators - 2 corresponding to fluxes and 2 corresponding to charges. (See Fig. 4).

Formally, we can write our low-energy Hamiltonian as

$$H'_{\text{eff}} = \sum_{\langle (\mathbf{p}, \mathbf{I})(\mathbf{q}, \mathbf{J}) \rangle} (t_{(\mathbf{p}, \mathbf{I})(\mathbf{q}, \mathbf{J})} + t_{(\mathbf{q}, \mathbf{J})(\mathbf{p}, \mathbf{I})})$$

where the hopping operators are defined by

$$t_{(\mathbf{p}, \mathbf{I})(\mathbf{q}, \mathbf{J})} = J_1 \sigma_i^1 \text{ or } J_3 \sigma_i^3$$

depending on whether $(\mathbf{p}, \mathbf{I}), (\mathbf{q}, \mathbf{J})$ differ by a flux hop or a charge hop. In the first case, \mathbf{i} is defined to be the link joining \mathbf{p} and \mathbf{q} , while in the second case, \mathbf{i} is the link joining \mathbf{I} and \mathbf{J} .

To calculate the statistics we need to compare a product of the form $t_1 t_2 t_3$ with the product $t_3 t_2 t_1$, where the t_k are hopping operators involving a single bound state at position (\mathbf{p}, \mathbf{I}) . (We could write out these expressions precisely, but it's messy and not very enlightening).

Now, as we discussed above, there are four different ways that a bound state at (\mathbf{p}, \mathbf{I}) can hop - two charge hops, and two flux hops. Each of the flux hopping operators can be paired with a corresponding charge operator which involves the same link \mathbf{i} . It's easy to see that each of the flux operators anti-commutes with the corresponding charge operator (since one involves a σ^1 , while the other involves a σ^3). However, everything else commutes. (See Fig. 4).

With these facts in mind, we can compare $t_1 t_2 t_3$ with $t_3 t_2 t_1$. By our discussion above, any set of 3 hopping operators involving (\mathbf{p}, \mathbf{I}) must contain exactly 2 which anti-commute. Thus, exactly 2 of t_1, t_2, t_3 anti-commute. This implies that

$$t_1 t_2 t_3 = -t_3 t_2 t_1$$

so that $e^{i\theta_{\text{stat.}}} = -1$. We conclude that the bound states are fermions.

Of course, this result is not that surprising once we notice that the charges and fluxes have relative statistics $e^{i\phi_{\text{rel.}}} = -1$.

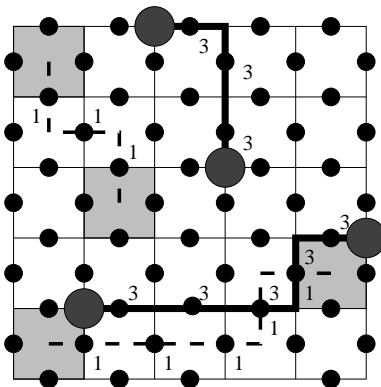


FIG. 5: Three examples of string-like creation operators. The string operators are drawn together with the particles they create at their endpoints. The thick solid line (drawn on the lattice) is an example of a charge string operator. The dotted line (drawn on the dual lattice) is an example of a flux string operator. At the bottom, we give an example of a string operator for the bound state of a charge and flux. It is essentially a combination of a charge string and a flux string.

We can easily check that the charges and fluxes have relative statistics -1 . Normally, we would just use the formula derived in the first section. However, this formula assumes that the two types of particles both hop on the same lattice. In our case, the charges hop on the square lattice, while the fluxes hop on the dual lattice.

It's not hard to see that in our case the formula needs to be modified to

$$t_{IJ}t_{pq} = e^{i\phi_{\text{stat.}}}t_{pq}t_{IJ}$$

where I, J are neighboring sites, p, q are neighboring plaquettes, and the links connecting I, J , and p, q are the same.

From our previous calculations we know that $t_{pq} = J_1\sigma_i^1$, and $t_{IJ} = J_3\sigma_i^3$. Therefore, t_{pq}, t_{IJ} anti-commute, and the relative statistics are $e^{i\phi_{\text{stat.}}} = -1$.

We now see why the two particles were called ‘‘charges’’ and ‘‘fluxes’’ - they have the same statistics and relative statistics as Z_2 charges and fluxes. It turns out that this connection with Z_2 gauge theory extends beyond the low energy regime - in fact, all the way to the lattice scale. One can show that the Kitaev model is exactly equivalent to standard Z_2 gauge theory coupled to a Z_2 Higgs field.

We argued earlier that whenever fermions or anyons occur in a bosonic system, they are always created in pairs, and the pair creation operator has a string-like structure. The above exactly soluble model (5) provides a good example of this phenomenon.

We begin with the charges. We can construct the string operators associated with these particles by taking products of their hopping operators along some path $P = I_1, \dots, I_n$ on the lattice. We find

$$\begin{aligned} W(P) &= t_{I_1 I_2} t_{I_2 I_3} \dots t_{I_{n-2} I_{n-1}} t_{I_{n-1} I_n} \\ &= \sigma_{i_1}^3 \sigma_{i_2}^3 \dots \sigma_{i_{n-2}}^3 \sigma_{i_{n-1}}^3 \end{aligned}$$

where $i_1 \dots i_{n-1}$ are the links along the path (see Fig. 5). Notice that $W(P)$ is nontrivial - it acts non-trivially on the bosonic degrees of freedom along the string P .

It is also a pair creation operator: If we apply $W(P)$ to the ground state, the resulting state is an exact eigenstate with two charges - one located at each endpoint of P . One way to see this is to notice that $W(P)$ commutes with everything in the Hamiltonian except $\prod_{C'_{I_1}} \sigma_j^1$ and $\prod_{C'_{I_n}} \sigma_j^1$. The string anti-commutes with these two operators, so when we apply it to the ground state, we get a state with two charges located at I_1, I_n .

The case of the flux quasi-particles is very similar. We take the product of the flux hopping operators along some path $P' = p_1, \dots, p_n$ on the dual lattice. We find

$$\begin{aligned} W'(P') &= t_{p_1 p_2} t_{p_2 p_3} \dots t_{p_{n-2} p_{n-1}} t_{p_{n-1} p_n} \\ &= \sigma_{i_1}^1 \sigma_{i_2}^1 \dots \sigma_{i_{n-2}}^1 \sigma_{i_{n-1}}^1 \end{aligned}$$

where $i_1 \dots i_{n-1}$ are the links along the path (see Fig. 5). Just as before, one can show that the string $W'(P')$ is a creation operator which creates two fluxes at the endpoints of P' .

Finally, consider the case of the bound state of the charge and the flux. The hopping operator for the bound state is a combination of the charge and flux hopping operators, so the string turns out to be a combination of the charge and flux strings. Let $P = (p_1, I_1) \dots (p_n, I_n)$ be a path in bound state configuration space. Then the associated bound state string operator is

$$W(P) = \prod_k \sigma_{i_k}^{a_k}$$

where $i_1 \dots i_{n-1}$ are the links along the path and $a_k = 1$ or 3 depending on whether (p_k, I_k) differ by a flux hop or a charge hop, respectively (see Fig. 5). Once again, one can show that the string $W(P)$ is a creation operator for a pair of bound states located at the ends of P .

In each of these examples, the creation operator for fermions or anyons has an extended string-like structure. It is important to note that the position of this string is completely unobservable. That is, in each case, the excited state $W(P)|0\rangle$ is independent of the position of P : it only depends on the position of the *endpoints* of P .

VI. EXACTLY SOLUBLE 3D MODEL

In two dimensional systems, one can create a fermion by binding a Z_2 vortex to a Z_2 charge. This is how we obtained the fermion in the above spin-1/2 model on the square lattice. Since both the Z_2 vortex and Z_2 charge appear as the ends of open strings, the fermions also appear as the ends of strings. However, in three dimensions, we cannot change a boson into a fermion by attaching a π -flux. Thus one may wonder if fermions still appear as the ends of strings in (3+1)D. In this section, we study an exactly soluble spin-3/2 model on a cubic lattice. We will show that the creation operator for fermions does indeed have a string-like structure. This example demonstrates that the string picture for fermions is more general than the flux-charge picture.

Our model has four states for each site of a cubic lattice. Thus we call it a spin-3/2 model. Let γ^{ab} ,

$a, b \in \{x, \bar{x}, y, \bar{y}, z, \bar{z}\}$, $a \neq b$, be 4×4 hermitian matrices that satisfy

$$\begin{aligned} \gamma^{ab} &= -\gamma^{ba} = (\gamma^{ab})^\dagger \\ [\gamma^{ab}, \gamma^{cd}] &= 0, \text{ if } a, b, c, d \text{ are all different} \\ \gamma^{ab}\gamma^{bc} &= i\gamma^{ac}, \quad a \neq c \\ (\gamma^{ab})^2 &= 1 \end{aligned} \quad (7)$$

A solution of the above algebra can be constructed by taking pairwise products of Majorana fermion operators λ^a , $a \in \{x, \bar{x}, y, \bar{y}, z, \bar{z}\}$:

$$\begin{aligned} \gamma^{ab} &= i\lambda^a\lambda^b \\ \{\lambda^a, \lambda^b\} &= 2\delta_{ab} \end{aligned} \quad (8)$$

The six Majorana fermion operators naturally require a space of dimension $2^{6/2} = 8$, but if we restrict them to the space $\prod_a \lambda^a = 1$, we obtain the desired 4×4 hermitian matrices. Alternatively, a more concrete description of the γ^{ab} is given in the appendix, where we express the γ^{ab} in terms of Dirac matrices.

In terms of γ_i^{ab} , the exactly soluble spin-3/2 Hamiltonian can be written as

$$H = -g \sum_{\mathbf{p}} F_{\mathbf{p}} \quad (9)$$

where \mathbf{p} labels all the square plaquettes in the cubic lattice, and the $F_{\mathbf{p}}$ are ‘‘flux’’ operators defined by

$$\begin{aligned} F_{\mathbf{p}} &= \gamma_i^{yx} \gamma_{i+\hat{x}}^{\bar{x}y} \gamma_{i+\hat{x}+\hat{y}}^{\bar{y}\bar{x}} \gamma_{i+\hat{y}}^{x\bar{y}} \\ \text{or } & \gamma_i^{zy} \gamma_{i+\hat{y}}^{\bar{y}z} \gamma_{i+\hat{y}+\hat{z}}^{\bar{z}\bar{y}} \gamma_{i+\hat{z}}^{y\bar{z}} \\ \text{or } & \gamma_i^{xz} \gamma_{i+\hat{z}}^{\bar{z}x} \gamma_{i+\hat{z}+\hat{x}}^{\bar{x}\bar{z}} \gamma_{i+\hat{x}}^{z\bar{x}} \end{aligned} \quad (10)$$

depending on the orientation of the plaquette \mathbf{p} . Just as in the Kitaev model,[15] all the $F_{\mathbf{p}}$ commute with each other and all the $F_{\mathbf{p}}$ have only two eigenvalues: ± 1 . Thus, we can solve the Hamiltonian by simultaneously diagonalizing all the $F_{\mathbf{p}}$. If $|\{f_{\mathbf{p}}\}\rangle$ is a common eigenstate of $F_{\mathbf{p}}$ with $F_{\mathbf{p}}|\{f_{\mathbf{p}}\}\rangle = f_{\mathbf{p}}|\{f_{\mathbf{p}}\}\rangle$, $f_{\mathbf{p}} = \pm 1$, then it is also an energy eigenstate with energy

$$E(\{f_{\mathbf{p}}\}) = -g \sum_{\mathbf{p}} f_{\mathbf{p}}$$

The one subtlety is that the $f_{\mathbf{p}}$ are not all independent. If C is the surface of a unit cube, then we have the operator identity

$$\prod_{\mathbf{p} \in C} F_{\mathbf{p}} = 1$$

It follows that

$$\prod_{\mathbf{p} \in C} f_{\mathbf{p}} = 1$$

for all cubes C . This constraint means that the spectrum of our model is identical to a Z_2 gauge theory on a cubic lattice. The ground state can be thought of as a state

with no flux: $f_{\mathbf{p}} = 1$ for all \mathbf{p} . Similarly, the elementary excitations are small flux loops where

$$f_{\mathbf{p}} = -1$$

for the four plaquettes \mathbf{p} adjacent to some link $\langle i\mathbf{j} \rangle$. We can think of these excitations as quasi-particles which live on the *links* of the cubic lattice.

We would like to compute the statistics of these excitations. It is tempting to assume that they are bosons since the model is almost the same as a Z_2 gauge theory. However, as we will show, this superficial similarity is misleading: the flux loops are actually fermions.

As in the Kitaev model, the quasi-particles are exact eigenstates of our Hamiltonian. Thus, they have no dynamics and it is difficult to compute their statistics. However, this is a special feature of our model. We need to perturb the theory to understand generic states in the same quantum phase. The simplest perturbation is

$$H' = H + t \sum_{i,a,b} \gamma_i^{ab}$$

To compute the statistics, we need to restrict ourselves to the low energy subspace with n quasi-particles. In this subspace, H' reduces to

$$H'_{\text{eff}} = t \sum_{i,a,b} \gamma_i^{ab}$$

The effect of each term γ_i^{ab} is to allow the quasi-particles to hop between the two links $\langle i(i+\hat{a}) \rangle$ and $\langle i(i+\hat{b}) \rangle$ adjacent to site i . Here $\hat{a} = \hat{x}$ if $a = x$, $\hat{a} = -\hat{x}$ if $a = \bar{x}$, etc. (See Fig. 6.) Thus, our Hamiltonian can be written in the standard hopping form:

$$H'_{\text{eff}} = \sum_{i,a,b} (t_{\langle i(i+\hat{a}) \rangle \langle i(i+\hat{b}) \rangle} + t_{\langle i(i+\hat{b}) \rangle \langle i(i+\hat{a}) \rangle})$$

where $t_{\langle i(i+\hat{a}) \rangle \langle i(i+\hat{b}) \rangle} = \frac{t}{2} \gamma_i^{ab}$.

To calculate the statistics we need to compare a product of the form $t_1 t_2 t_3$ with the product $t_3 t_2 t_1$, where the t_k are hopping operators involving a single link $\langle i\mathbf{j} \rangle$. (As in the Kitaev model, we could write out these expressions explicitly, but it's not very enlightening).

Note that a quasi-particle on the link $\langle i\mathbf{j} \rangle$ can hop to any of the 10 neighboring links, 5 of which are adjacent to site i , and 5 of which are adjacent to site j (see Fig. 6). Using the algebra (7) or the Majorana fermion representation, we find that the 5 hopping operators associated with site i all anti-commute with each other and similarly for site j . On the other hand, each of the operators associated with i commute with each of the operators associated with j .

With these facts in mind, we can now compare $t_1 t_2 t_3$ with $t_3 t_2 t_1$. There are essentially two cases: either all three of the t_k 's are associated with a single site i or j , or two involve one site and one involves the other. In the first case, all the t_k 's anti-commute; in the second case, a pair of the t_k 's anti-commute and everything else commutes. In either case, we have

$$t_1 t_2 t_3 = -t_3 t_2 t_1$$

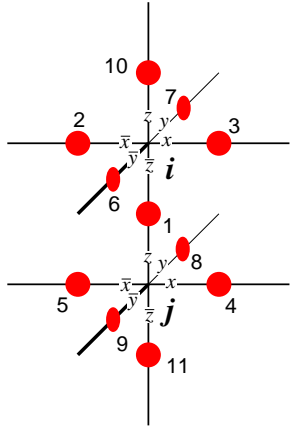


FIG. 6: A fermion on link 1 can hop onto 10 different links: 2 - 11. The hopping $1 \rightarrow 2$ is generated by $\gamma_i^{\bar{x}\bar{z}}$, the hopping $1 \rightarrow 10$ by $\gamma_i^{z\bar{z}}$, and the hopping $1 \rightarrow 4$ by γ_j^{xz} .

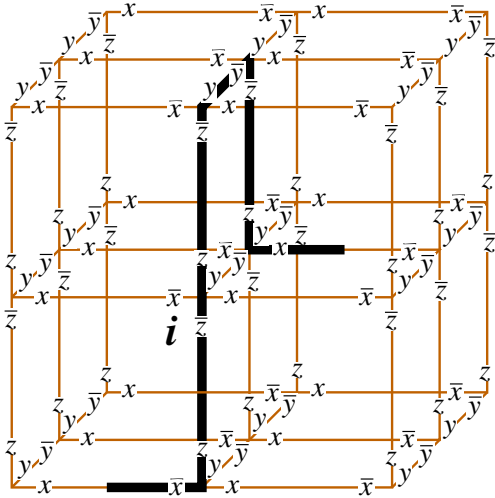


FIG. 7: An example of a string in the 3D model. This string corresponds to the operator $W(P) = \gamma_{i-\bar{z}}^{\bar{x}\bar{z}} \gamma_{i-\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{\bar{y}\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}} \gamma_{i+\bar{z}}^{z\bar{z}}$.

We conclude that the quasi-particles are indeed fermions.

Next, we construct the associated string operator. Taking a product of hopping operators along a path (of links), $P = \langle i_1 i_2 \rangle, \langle i_2 i_3 \rangle, \dots, \langle i_{n-1} i_n \rangle$, gives

$$W(P) = \gamma_{i_1}^{a_1 b_1} \gamma_{i_2}^{a_2 b_2} \dots \gamma_{i_n}^{a_n b_n} \quad (11)$$

where b_m, a_{m+1} is the pair of indices associated with the link $\langle i_m i_{m+1} \rangle$. (See Fig.7.) Notice that the string operator commutes with the Hamiltonian, except at the ends. Thus, when we apply $W(P)$ to the ground state, the only effect is to flip the signs of $f_{\mathbf{p}}$ for the plaquettes \mathbf{p} adjacent to the links $\langle i_1 i_2 \rangle, \langle i_{n-1} i_n \rangle$. This means that the open string operator creates a pair of fermions at its two ends.

VII. CONCLUSION

It is well-known that gauge theories and strings are closely related.[23–29] In this paper, we have shown that there is also a close connection between strings and fermions. It appears that the fundamental concepts of Fermi statistics, gauge theory, and strings are all intimately related. They are just different aspects of a new kind of order - topological order - in bosonic lattice systems.

Perhaps more importantly, we have derived a general relation (2) between the statistics of a lattice particle and the algebra of its hopping operators. Since the hopping operators for emerging fermions or anyons are determined by the associated strings, we can use (2) to determine the statistics of the ends of strings from the structure of the string operator. In this way, the statistical algebra is the foundation of the string picture of fermions or anyons.

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APPENDIX A: DIRAC MATRIX REPRESENTATION OF γ^{ab}

In the following, we represent γ^{ab} 's in terms of the Dirac matrices. Note that Eq. (7) implies that

$$\{\gamma^{ac}, \gamma^{bc}\} = 2\delta_{ab}$$

Thus $\gamma^{xz}, \gamma^{\bar{x}\bar{z}}, \gamma^{yz},$ and $\gamma^{\bar{y}\bar{z}}$ satisfy the algebra of Dirac matrices. Introducing four Dirac matrices

$$\begin{aligned} \gamma^x &= \sigma^1 \otimes \sigma^1, & \gamma^{\bar{x}} &= \sigma^2 \otimes \sigma^1 \\ \gamma^y &= \sigma^3 \otimes \sigma^1, & \gamma^{\bar{y}} &= \sigma^0 \otimes \sigma^2, \end{aligned}$$

we can write

$$\gamma^{az} = \gamma^a, \quad a = x, \bar{x}, y, \bar{y}. \quad (A1)$$

Similarly, $\gamma^{x\bar{z}}, \gamma^{\bar{x}\bar{z}}, \gamma^{y\bar{z}},$ and $\gamma^{\bar{y}\bar{z}}$ satisfy the algebra of Dirac matrices. We can express those operators as

$$\gamma^{a\bar{z}} = i\gamma^a \gamma^5, \quad a = x, \bar{x}, y, \bar{y}. \quad (A2)$$

where $\gamma^5 = \gamma^x \gamma^{\bar{x}} \gamma^y \gamma^{\bar{y}}$. Finally, for $a, b = x, \bar{x}, y, \bar{y}$, we have

$$\gamma^{ab} = i\gamma^a \gamma^b \quad (A3)$$

In this way, we can express all the γ^{ab} , $a, b \in \{x, \bar{x}, y, \bar{y}, z, \bar{z}\}$, $a \neq b$ in terms of Dirac matrices.

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