

**Many-body Systems with Non-Abelian Statistics** \**B. Blok*

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ABSTRACT: We use the conformal field theory to construct many-body Hamiltonians whose ground states support quasiparticle excitations with the non-abelian statistics. The explicit forms of the many-body ground-state wave functions are calculated. The above systems may appear naturally in the Fractional Quantum Hall Effect (FQHE) in the low density limit. We discuss the calculation of the quasiparticle statistics from the low energy effective theory and from the microscopic wave function. The stability of the non-abelian statistics under small perturbations of the system is also considered. We argue

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that the non-abelian statistics is a topological property of the many-body systems. This property does not depend on the detailed short-ranged structures of the systems.

## 1. INTRODUCTION.

Recently there was a considerable interest in studying the systems with the fractional statistics in 2+1 dimensions.<sup>1,2</sup> Such systems appear in the fractional quantum Hall effect<sup>3</sup> and in some hypothetical models of superconductivity.<sup>4</sup>

The systems with the fractional statistics are closely related to the quantum field theory with the action given by the Chern-Simons (C-S) term, in the sense that the C-S theory describes the long range behavior of such systems. The construction of many-body systems with the fractional statistics can be helpful if we wish to study the properties of the C-S theory. This is especially important since the non-abelian C-S theory is not quasiclassical due to the presence of the cubic term  $f^{abc}\epsilon_{\mu\nu\lambda}A_\mu^a A_\nu^b A_\lambda^c$  in the C-S action. This term causes the main difficulty in studying the C-S theories, although there exist some methods that allow one to deal with it beyond the perturbation theory.<sup>5</sup> The many-body systems with the non-abelian statistics considered in Ref. 6 and in this paper can be useful in giving an alternative approach to study the C-S theory. These many-body systems provide a natural ultraviolet cut off to the non-abelian C-S theory. The connection between the U(1) C-S theory and the many-body systems has appeared already in the theory of the Laughlin<sup>7,8,9</sup> and the hierarchical FQH states.<sup>10,11,12</sup>

The application to the Fractional Quantum Hall effect is one of the most important physical applications of the fractional statistics. It becomes more and more clear that the FQHE and the C-S theory are closely related. Many recent studies suggest that the incompressible FQH states are classified by the different C-S theories. For example the hierarchical FQH states are described by the U(1) C-S theories that may contain several gauge fields.<sup>11,12,10</sup> Consequently the hierarchical FQH states are classified by the integer valued symmetric matrices that appear as the coefficients in the C-S term. These systems have the abelian statistics.

It is natural to ask whether the non-abelian C-S theories are related to the FQHE. Recently it was shown in Ref. 6 that the FQH states described, for example, by the wave functions  $[\chi_N(\{z_i\})]^k$  are related to the non-abelian CS theories. Here  $\chi_N$  is the fermion wave function of  $N$  filled Landau levels, and  $z_i$  are the electron coordinates. The effective theory of such a FQH state was shown to be the  $SU(k)$  level  $N$  CS theory, and the charge  $e/k$  quasiparticles were shown to carry the non-abelian statistics. Physically this means two things: A) In the presence of several quasiparticles, the lowest energy states are almost degenerated even when the positions of the quasiparticles are fixed. The energy differences between these degenerated low lying states approach zero as the separation between the quasiparticles goes to infinity. B) If we interchange two identical quasiparticles, these degenerated low lying states will transform through each other (acquiring the non-abelian Berry's phases<sup>13</sup>). The matrices of the transformations for different interchanging paths form a representation of the braid group. (In the abelian case the low lying states are not degenerated and the wave functions are only changed by a phase if we interchange the identical particles).

The simplest non-abelian FQH state (the NAF state) is the  $\nu = \frac{1}{2}$  state described by the wave function  $\chi_1(\chi_2)^2$ . The quasiparticles that carry the charge  $e/4$  have the non-abelian statistics described by the  $SU(2)$  level 2 CS theory.

The above NAF states are the ground states of some local Hamiltonians with short range repulsive interactions. Therefore there are real possibilities to observe the NAF states in the experiments.

In this paper we plan to do two things. A) To generalize the construction of Ref. 6 and to obtain more general non-abelian FQH states. B) To try to understand and calculate the non-abelian statistics from the microscopic electron wave functions.

Let us first consider the problem B). The calculations in Ref. 6 are based on the mean-field effective field theory. It is desirable to confirm these results from a microscopic calculation based on the electron wave function. One way to calculate the statistics of the quasiparticles is to relate the electron wave function of the FQH state to certain correlation functions (which are called conformal blocks) in conformal field theory and use the known results on braiding matrices of the conformal blocks to calculate the statistics. The idea to use the conformal field theory (CFT) to describe the electron wave functions was first proposed in Ref. 14.

But there is one difficulty. The wave functions of the NAF states,  $[\chi_N(\{z_i\})]^k$ , contain the components in the higher Landau levels. These wave function are not holomorphic even after excluding the Gaussian factor. Thus we cannot directly relate the wave function  $[\chi_N(\{z_i\})]^k$  to the correlation function in the conformal field theory. The above problem can be partially resolved in the following way. Notice that when  $N = 2$ , the electrons in the state  $[\chi_2(\{z_i\})]^k$  lie in first  $k + 1$  Landau levels. We may treat the Landau level index as a “flavor” index. Instead considering one specie of electrons in the first  $k + 1$  Landau levels, we may consider “electrons” which carry  $k + 1$  flavors but now are in the first Landau level. These  $k + 1$  flavors form a “spin”  $k/2$  representation of  $SU(2)$  group. For the convenience we will call these flavored electrons  $SU(2)$  electrons or simply electrons. The wave function  $[\chi_2(\{z_i\})]^k$  can be mapped into the wave function of the  $SU(2)$  electrons with all the electrons in the first Landau level (see section 2.B). Such a wave function has a form  $f(z_i, m_i)e^{-\frac{1}{4}\sum_k |z_k|^2}$  where  $m_i = 1, \dots, k + 1$  are the  $SU(2)$  flavor indices of the  $i^{th}$  electron. The prefactor  $f(z_i, m_i)$ , being a holomorphic function, can be expressed as a correlation function in the  $SU(2) \times U(1)$  Kac-Moody (KM) algebra. Thus it would be interesting to consider a system of  $SU(2)$  electrons and to see whether the results obtained from the mean field effective theory can be confirmed by microscopic calculations.

However the above results do not apply for  $N > 2$ . This is because the electrons in  $[\chi_N(\{z_i\})]^k$  lie in first  $k(N - 1) + 1$  Landau levels which naturally lead to electrons with  $k(N - 1) + 1$  flavors. But for  $N > 2$ ,  $k(N - 1) + 1$  is in general not a dimension of representations of  $SU(N)$  group. Nevertheless, it is possible to construct the microscopic systems for the  $SU(N)$  electrons so that the ground states are described by the holomorphic wave functions and the low lying excitations have the  $SU(N)$  statistics. Certainly these wave functions do not have the straightforward Landau level interpretation as in the  $SU(2)$  case.

In the real experimental situations the flavor index of the electrons can arise in many ways. The flavor index may come from the single particle bands associated with higher Landau levels, spins, higher subbands in the quantum well and/or layers of the 2D electron gas. When the Coulomb interaction between electrons is larger than the energy spacing between different bands (say in a weak field limit), it is not proper to consider only the first Landau level with the polarized spins. The ground state should be constructed from the states in several low lying bands.

In this paper we will consider the following special idealized situation: A) There are  $K$  single-particle bands which are closely degenerated B)  $K$  is a dimension of a representation of  $SU(N)$  group (or some other group) C) The Hamiltonian has an (approximated)  $SU(N)$  symmetry. In this case we can treat the band index as the  $SU(N)$  index and regard the electrons as the  $SU(N)$  electrons.

Motivated from the above discussion, in this paper we will study systems of  $SU(N)$  electrons. We will construct the many-body states of the  $SU(N)$  electrons in magnetic field in which the elementary excitations have non-abelian statistics. We will also construct local Hamiltonians (which respect  $SU(N)$  symmetry) such that our many-body states are their exact ground states. The non-abelian statistics is calculated from the low energy effective theory and from the microscopic wave functions. We also argue that the degeneracy required by the non-abelian statistics is robust against any small perturbations of the system, even against those that break the  $SU(N)$  symmetry. This implies that the fractional statistics is a topological quantity that is independent from the details of the model. The long range properties of our model are described by the same non-abelian C-S theory independent of the nature of the perturbation.

Finally we would like to remark that in real experimental situations different bands are not equivalent and we do not have exact  $SU(N)$  (or other group) symmetries. The electrons in different bands (*i.e.*, electrons with different flavors) in general have different energies. In our construction the different flavors are treated on equal footing. The wave functions constructed in the next section respect the flavor symmetry (*i.e.*, the  $SU(N)$  symmetry) and may not be good trial wave functions for the ground state for real cases. However if the experimental situation is such that energy difference between a few bands is less than the energy of electron interaction and the Hamiltonian has an approximate  $SU(N)$  symmetry, then our wave functions may represent the universality classes of the true ground states. In this case we may have some real chance to observe the NAF states experimentally.

The paper is organized as follows. In the section 2 we describe in detail the construction of the wave functions of the ground state and the low energy excited states of our model. We shall use three different approaches to the WZW model: fermionization<sup>15,16</sup>, non-abelian bosonization<sup>17</sup> and the connection with the parafermions<sup>18</sup> (this connection will be described in details in the appendix B). We also construct local Hamiltonians, such that our wave functions are the exact ground states of the Hamiltonians. In section 3 we calculate the non-abelian statistics directly from the many-body wave functions. The analogous calculation in the abelian case was made in Ref. 19,11,12. We assume that the NAF state is incompressible. In section 4 we consider small but otherwise arbitrary perturbations of our model Hamiltonian. We argue that the non-abelian statistics is still well defined for the perturbed Hamiltonian. The degeneracy associated with the non-abelian statistics can not be lifted by any perturbations. In section 5 we summarize our results. In particular we discuss the assumptions that we made in order to obtain our results. Some useful facts about the Kac-Moody algebras, WZW modes and their connection to the parafermions are given in the appendices A and B. In the appendix C we consider the Feigin-Fuchs type representation of the wave functions.

## 2. THE CONSTRUCTION OF THE ELECTRON SYSTEM.

In this section we shall describe explicitly many-body systems of the  $SU(N)$  electrons which have the non-abelian statistics. We shall use the  $SU(N) \times U(1)$  Kac-Moody (K-M) algebra to construct the wave functions of the  $SU(N)$  electrons in the first Landau level. The wave functions of the ground state and of the quasiparticle excitations are represented by the conformal blocks of the corresponding  $SU(N) \times U(1)$  model. We also construct local hamiltonians for which our wave functions are the exact ground states.

## 2.a Conformal Field Theory and the Construction of the Wave Functions.

Let us begin from the construction of the wave functions. In order to construct the wave functions we shall use the Wess-Zummino-Witten model. (We refer the reader to the appendix A for some useful facts about the WZW model that we shall use in this section). The two dimensional correlation functions of the primary fields in the 2d WZW model can be represented as the sums of the squares of the holomorphic functions called the conformal blocks:

$$\left\langle \prod_{i=1}^{i=n} V_{\lambda_i}^{\Lambda_i}(z_i) \bar{V}_{\lambda_i}^{\Lambda_i}(\bar{z}_i) \right\rangle = \sum_{\alpha} |F_{\alpha}^{\Lambda_1, \dots, \Lambda_n}(z_1, \lambda_1, \dots, z_n, \lambda_n)|^2 \quad (1)$$

The indices  $\Lambda_i$  denote the highest weights of the representation of the  $SU(N)$  Lie algebra that labels the primary fields  $V_{\lambda_i}^{\Lambda_i}$  of the WZW model. The index  $\lambda_i$  is the weight of the  $SU(N)$  representation with the highest weight  $\Lambda_i$ . The index  $\alpha$  is the additional quantum number that labels the different conformal blocks. The functions  $F_{\alpha}$  form the basis of the space of conformal blocks that corresponds to the specific choice of the intermediate channels in the Green functions (1). All conformal blocks are holomorphic functions.

Under the interchange of the two particles the functions  $F_{\alpha}$  transform between each other:<sup>20</sup>

$$\begin{aligned} F_{\alpha}^{\Lambda_1, \dots, \Lambda_n}(\dots, z_i, \lambda_i, z_{i+1}, \lambda_{i+1}, \dots) &= \\ &= \sum B_{\alpha\beta} F_{\beta}^{\Lambda_1, \dots, \Lambda_n}(\dots, z_{i+1}, \lambda_{i+1}, z_i, \lambda_i, \dots) \end{aligned} \quad (2)$$

The exchange matrices  $B$  form the representation of the Braid group and are *independent* of the  $SU(N)$  group indices  $\lambda_i$ . (See Ref. 21 for a detailed discussion about the connection between the braid groups and the conformal field theory). The space of the conformal blocks for a given set of the primary fields is in the one to one correspondence with the set of the invariant tensors of the  $SU(N)$  quantum group.<sup>22</sup>

The conformal blocks in the WZW model have a very important property that they satisfy the Knishnik-Zamolodchikov (K-Z) equation:<sup>23</sup>

$$\partial_{z_i} F_{\alpha}^{\Lambda_1, \dots, \Lambda_n}(\{z_l, \lambda_l\}) + \frac{1}{k + c_V} \sum_{j \neq i; \lambda'_i, \lambda'_j} \frac{T_{\lambda_i \lambda'_i}^{\Lambda_i, a} T_{\lambda_j \lambda'_j}^{\Lambda_j, a}}{z_i - z_j} F_{\alpha}^{\Lambda_1, \dots, \Lambda_n}(\{z_l, \lambda'_l\}) = 0 \quad (3)$$

where  $k$  is the level of the WZW model,  $c_V$  is a Casimir operator, ( $c_V = N$  for  $SU(N)$ ) and  $T^{\Lambda_i, a}$  are the generators of the Lie group in the representation with the highest weight  $\Lambda_i$ .  $T^{\Lambda_i, a}$  satisfy the equation  $tr(T^a T^b) = 2\delta^{ab}$ . The K-Z equation completely determines the conformal blocks (with suitable boundary conditions). Thus the K-Z equation can be regarded as the defining condition of the conformal blocks.

We will attempt to express the wave functions of the  $SU(N)$  electrons in terms of the WZW conformal blocks. Assume that the  $SU(N)$  electrons are in a representation of highest weight  $\Lambda$ . We can try to express the electron wave function as the following conformal blocks in the  $SU(N)_k \times U(1)$  WZW model: (cf. Ref. 14)

$$\Psi_{\alpha}(z_1, \lambda_1, \dots, z_n, \lambda_n) = \left\langle \prod_{i=1}^{i=n} V_{\lambda_i}^{\Lambda}(z_i) \exp(iq\phi(z_i)) \exp\left(-\int \phi(z') d^2 z'\right) \right\rangle_{\alpha} \quad (4)$$

The operators  $\exp(iq\phi(z_i))$  are the vertex operators of the  $U(1)$  model and  $V_{\lambda_i}^\Lambda(z_i)$  are the vertex operators of the  $SU(N)_k$  WZW model. The last factor in eq. (4) corresponds to the interaction with the uniform neutralizing background. The value of  $q$  will be determined later. Notice that the chiral correlation function (4) is not completely determined by the operators. The ambiguity corresponds to the existence of the different conformal blocks that are labeled by  $\alpha$ . The right hand side of eq. (4) is (by definition) an alternative notation for the conformal blocks defined through eq. (1).

It is pointed out in Ref. 14 that the last multiplier in the eq. (4) gives rise to the gaussian factor. Since  $U(1)$  and  $SU(N)$  are decoupled, the wave functions have the form

$$\Psi_\alpha(z_1, \lambda_1, \dots, z_n, \lambda_n) = F_\alpha^\Lambda(z_1, \lambda_1, \dots, z_n, \lambda_n) F^{U(1)}(z_1, \dots, z_n) \prod_{i=1}^{i=n} \exp(-q|z_i|^2) \quad (5)$$

Here  $F_\alpha$  is the conformal block of the  $SU(N)$  model defined by eq. (1) and  $F^{U(1)}$  is the conformal block of the  $U(1)$  model which has the form:

$$F^{U(1)}(z_1, \dots, z_n) = \prod_{i,j=1}^{i,j=n} (z_i - z_j)^{q^2} \quad (6)$$

Let us now discuss the physical conditions on the conformal blocks so that the wave function defined in eq. (4) can be regarded as an electron wave function. The conformal blocks in eq. (4) should satisfy three main conditions in order to be the electron wave functions. A) They must be single valued antisymmetric holomorphic functions of  $z_i$  and  $\lambda_i$ , so that they describe electron states in the first Landau level. B) They must not contain any poles or other infinities. C) They must have a well defined thermodynamic limit. This means that the wave function are well defined for large numbers of the electrons ( $n \rightarrow \infty$ ).

The condition A) and C) require that the vertex operators  $V_\lambda^\Lambda$  must a simple current. The simple current is a primary field which has an unique fusion rule with any other primary fields. This means that the operator product expansion between the simple current and any other primary fields in the theory contains only one primary field. Therefore the braiding matrices between a simple current and any other primary fields are always abelian. In this case by properly choosing  $q$  in (4) we can make the wave function to be a single valued antisymmetric function. If  $V_\lambda^\Lambda$  was not a simple current, the number of the conformal blocks would diverge as  $n \rightarrow \infty$ . In this case the wave function in general can never be single valued no matter how we choose  $q$  since the braiding matrices are non-abelian. The conformal blocks that satisfy A) and C) have already been considered in conformal field theories in completely different contexts.<sup>24,25,26</sup> The name simple current is adopted following Ref. 24.

It appears that the above conditions are highly restrictive. They almost uniquely fix the choice of the primary fields in the WZW model that can be used as the electron operator. The full classification of the simple currents was made in Ref. 26,25. In the case of the  $SU(N)$  Kac-Moody algebra that we consider here, the simple currents correspond to the vertex operators in the representations of the highest weights  $k\Lambda^l$ . Here  $\Lambda^l$  ( $l = 1, \dots, N-1$ ) are the fundamental weights of the  $SU(N)$  algebra. These representations are given by the Young schemes in Fig. 1.

Next let us use the condition A) and B) to determine the  $U(1)$  part of the electron operator. Note that the conformal block of the simple currents is not really single valued. If we interchange the positions of two simple-current operators, the conformal block acquires a phase. We choose the  $U(1)$  part of the electron creation operator to make the total wave function single valued. The orders of poles in  $F^{k\Lambda^l}$  can be calculated from the operator product expansion of the simple currents

$$V^{k\Lambda^{l_1}}(z_1)V^{k\Lambda^{l_2}}(z_2) \rightarrow \frac{1}{(z_1 - z_2)^{h_{l_1+h_{l_2}}-h_{[l_1+l_2]}}} V^{k\Lambda^{[l_1+l_2]}} \quad (7)$$

$$h_l = \frac{kl(N-l)}{2N}, \quad l = 1, \dots, N$$

where  $[l] = l \bmod N$  and satisfies  $1 \leq [l] \leq N$ . we find that the conformal block satisfies

$$F^{k\Lambda^l}(z_1, \lambda_1, \dots, z_n, \lambda_n) \rightarrow (z_i - z_j)^{-q_0^2} \quad (8)$$

$$q_0^2 = h_{[2l]} - 2h_l$$

as  $z_i \rightarrow z_j$ . Thus to make (4) single valued we must choose

$$q^2 = l^2k/N + p \quad 2l \leq N \quad (9)$$

$$q^2 = (l - N)^2k/N + p \quad 2l \geq N$$

where  $p$  is an integer. The integer  $p$  must be non-negative so that the wave function contains no poles.

In the above we only showed that the eq. (9) is a necessary condition for the wave function to be single valued. Now we would like to show it is sufficient, and with proper choice of  $p$  the wave function can also be made antisymmetric. We know the ground state electron wave function is obtained from the correlation function of the electron operators:

$$V_{el, \lambda_i}^l = \exp(iq\phi(z_i))V_{\lambda_i}^{k\Lambda^l} \quad (10)$$

where  $q$  satisfies eq. (9). Let us now show that this wave function is single valued. Indeed, according to the fermion-Bose equivalence discussed in Ref. 16,27, the vertex operators (the simple currents) that we used are related to the composite fields made of chiral fermions in the following way:

$$\exp(iq_0\phi(z))V_{\lambda}^{k\Lambda^l}(z) \propto G_{\lambda}^l \quad (11)$$

where  $q_0$  is given by (8) and

$$G_{\lambda}^l = f_{\lambda_1, \dots, \lambda_k}^{\lambda} \psi_{\lambda_1}^{(1)} \dots \psi_{\lambda_k}^{(k)} \quad (12)$$

The fields  $\psi_{\lambda}^{(\alpha)}$  are fermionic field in the representation of highest weight  $\Lambda^l$ , and the tensor  $f^{\lambda}$  is the relevant Clebsch-Gordan (C-G) coefficient. When  $\Lambda^l = \Lambda^1$ ,  $f^{\lambda}$  is the symmetric tensor. The electron wave function now has a form

$$\Psi^l = \prod (z_i - z_j)^p \langle \prod G_{\lambda_i}(z_i) \rangle \prod \exp(-q|z_i|^2) \quad (13)$$

Because  $G_\lambda$  is formed by fermion operators, it is easy to see that  $\Psi^l$  is single valued. It is clear that  $G_\lambda$  is fermionic (bosonic) if  $k$  is odd (even). Thus the electron wave function is antisymmetric if  $k + p$  is an odd integer.

We conclude that the electron wave functions can be constructed from the correlation functions of simple current. For the  $SU(N)$  electrons, the ground state wave function has the form:

$$\begin{aligned}\Psi^l(z_1, \lambda_1, \dots, z_n, \lambda_n) &= \langle \prod V_{el, \lambda_i}^l(z_i) \exp(-i \int \phi d^2 z) \rangle \\ &= F^{k\Lambda^l}(z_1, \lambda_1, \dots, z_n, \lambda_n) \prod (z_i - z_j)^{q^2} \prod \exp(-1/4|z_i|^2)\end{aligned}\quad (14)$$

Let us calculate the filling fraction of the state that is described by the wave function (14). Notice that the total angular momentum carried by  $F^{k\Lambda^l}$  is of the order  $n$ . Thus the wave function  $\Psi^l$  has the total angular momentum

$$L = q^2 \frac{n(n-1)}{2} + O(n) \quad (15)$$

From the relation  $L = \frac{1}{\nu} \frac{n(n-1)}{2}$ , we find the filling fraction of the state (14) is given by

$$\nu = q^{-2} = \frac{N}{kl^2 + Np}. \quad (16)$$

We can now consider the wave functions of the excited states. These wave functions are obtained by insertion of arbitrary  $SU(N)_k$  vertex operators, multiplied by appropriate  $U(1)$  operators  $\prod (z_i - \eta_I)^{q_I q}$ , into the correlator (14). The wave functions have the form:

$$\begin{aligned}\Psi(z_i, \lambda_i, \eta_I, a_I, \alpha) &= \langle \prod_i V_{el, \lambda_i}^l(z_i) \prod_I e^{iq_I \phi(\eta_I)} V_{a_I}^{\Lambda_I}(\eta_I) e^{-i \int \phi d^2 z} \rangle_\alpha \\ &= \langle \prod_i V_{\lambda_i}^{k\Lambda^l}(z_i) \prod_I V_{a_I}^{\Lambda_I}(\eta_I) \rangle_\alpha \\ &\quad \prod_{i < j} (z_i - z_j)^{q^2} \prod_{I < J} (\eta_I - \eta_J)^{q_I q_J} \prod_{I, i} (z_i - \eta_I)^{q q_I} \prod_I e^{-q_I |\eta_I|^2} \prod_i e^{-q |z_i|^2}\end{aligned}\quad (17)$$

Here  $\eta_I$  and  $a_I$  are the positions and the  $SU(N)$  quantum number of the quasiparticles and  $\alpha$  is the conformal block index. In eq. (17)  $q_I$  is chosen so that the wave function is a single valued function of the electron coordinates  $z_i$ . The value of  $q_I$  can be found in the same way as the value of  $q$  was found above. For the case of the quasiparticle in the fundamental representation and  $p=0$  in eq. (9) it is equal to

$$q_I = \frac{h_{sf} - h_s - h_f}{q} = \frac{l}{qN} \quad (18)$$

Here  $h_s$  is the conformal dimension of the simple current,  $h_f$  is the conformal dimension of the field in the fundamental representation and  $h_{sf}$  is the conformal dimension of the

primary field that appears in the operator product of the simple current and the primary field in the fundamental representation.

Note that there are several wave functions that correspond to the insertion of the same vertex operators. These different wave functions correspond to the different conformal blocks that appear in the correlation function of the given set of the vertex operators. The simplest nontrivial example is the insertion of the four vertex operators: two  $V^{\Lambda^1}$  and two  $V^{\bar{\Lambda}^1}$ , where  $\bar{\Lambda}^1$  is the conjugate of  $\Lambda^1$ . In this case we have two conformal blocks that correspond to the symmetric and antisymmetric representations in the intermediate channels. As we will see later that all wave functions that correspond to the different choice of the conformal blocks have the same value of energy for certain local Hamiltonians. The different wave functions (*i.e.*, the conformal blocks) is in one to one correspondence with the tensors invariant under the  $SU(N)$  quantum group at level  $k$ .<sup>22</sup>

In this paper we shall use three different descriptions of the wave functions based on three different representations of the WZW conformal blocks: the the parton representation (next subsection), parafermionic representation (appendix B) and Feigin-Fuchs representation (Appendix C).

## 2.b Parton Construction and Low Energy Effective Theory

In the following we will use the parton construction to construct the wave function of the  $SU(N)$  electrons.<sup>6,16,28</sup> The parton construction has been used in Ref. 28 to calculate the 4-point correlation function of the simple currents. It has also been used to construct the hierarchical FQH states.<sup>29,30</sup> The parton construction is useful in two ways. A) It gives a simple explicit form of the ground state wave function. B) It is closely related to the mean field theory developed in Ref. 6. The statistics of the quasiparticle excitations can be easily calculated from the effective theory.

First, as a simple example, let us consider a wave function  $(\chi_2)^k$  for the spinless electrons. The wave function  $(\chi_2)^k$  can be constructed from the parton construction by first splitting each electron into  $k$  different fermions that are called partons, then letting each kind of partons to form a  $\nu = 2$  IQH state described by the wave function  $\chi_2$ . The electron wave function is obtained by recombining  $k$  partons together. This procedure leads to the wave function  $(\chi_2)^k$ . Note that the electrons in the wave function occupy the first  $k + 1$  Landau levels. In order to obtain a holomorphic wave function, we may regard the Landau level index as an “spin” index (of a  $SU(2)$  representation). Instead of considering the spinless electrons in the first  $k + 1$  Landau levels, we can equivalently study the spin- $\frac{k}{2}$  electrons in the first Landau level. The wave function of the spin- $\frac{k}{2}$  electrons is holomorphic (up to the Gaussian factors) and can be expressed as a correlation function of the conformal field theory.

To obtain the explicit form of the spin- $\frac{k}{2}$  electron wave function related to the wave function  $(\chi_2)^k$ , we may again use the parton construction. We first split the spin- $\frac{k}{2}$  electron  $\psi^m$  into  $k$  kinds of spin- $\frac{1}{2}$  fermions  $f^{a_i}$  (*i.e.*, the partons):

$$\begin{aligned} \psi^m &= f^{1a_1} \dots f^{ka_k} S_{a_1 \dots a_k}^m \\ m &= -\frac{k}{2}, -\frac{k}{2} + 1, \dots, \frac{k}{2} \quad a_i = \pm \frac{1}{2} \end{aligned} \tag{19}$$

Here  $m$  and  $a, b$  are the spin indices for the electrons and the partons. The coefficients  $S_{a_1, \dots, a_k}^m$  form the basis of the rank  $k$  symmetric tensors. When  $k = 2$ ,  $S_{a_1, a_2}^m$  are given by  $2 \times 2$  matrices

$$S^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S^0 = \begin{pmatrix} 0 & 2^{-\frac{1}{2}} \\ 2^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (20)$$

We then let the spin  $\frac{1}{2}$  partons fill the first Landau level. Such a state is described by the parton wave function  $\chi_s(z_i, a_i)$  which is a spin singlet. The spin- $\frac{k}{2}$  electron wave function can be constructed from the parton wave functions by recombining the  $k$  kinds of spin- $\frac{1}{2}$  partons into the spin- $\frac{k}{2}$  electrons. The above construction leads to the following electron wave function

$$\Psi(z_i, m_i) = \sum_{a_i^1, \dots, a_i^k} S_{a_i^1, \dots, a_i^k}^{m_i} \chi_s(z_i^1, a_i^1) \dots \chi_s(z_i^k, a_i^k) |_{z_i^1 = \dots = z_i^k = z_i} \quad (21)$$

$\chi_s(z_i^\alpha, a_i^\alpha)$  is the wave function of the  $\alpha^{\text{th}}$ -type partons.  $\Psi$  is also a spin singlet. Notice that the parton wave function  $\chi_s$  has a filling fraction  $\nu = 2$  and corresponds to the wave function  $\chi_2$  of the spinless fermions. We expect that the wave function in eq. (21) corresponds to the wave function  $(\chi_2)^k$  and has non-abelian statistics. This is indeed true. We can show, by generalizing the results in Ref. 6, that the effective theory of the state (21) is also described by the  $SU(k)$  level 2 non-abelian CS theory. Therefore the wave function (21) also supports quasiparticles with non-abelian statistics of  $SU(k)$  level 2.

The above discussion can be easily generalized to the  $SU(N)$  electrons which form a representation of  $SU(N)$  with highest weight  $k\Lambda^1$  (*i.e.*, the representation of the rank  $k$  symmetric tensor). In the parton construction we split the  $SU(N)$  electron (described by  $\psi^m$ ) into  $k$  kinds of partons in the fundamental representation:

$$\psi^m = f^{\alpha_1 a_1} \dots f^{\alpha_k a_k} \epsilon_{\alpha_1 \dots \alpha_k} S_{a_1 \dots a_k}^m \quad (22)$$

where  $\alpha$  are the ‘‘color’’ index which label different partons and  $S_{a_1 \dots a_k}^m$  is a basis of rank  $k$  symmetric tensors. Now  $m$  labels the states in the  $k\Lambda^1$  representations and  $a = 1, \dots, N$  labels the states in the fundamental representation  $\Lambda^1$ . The first-Landau-level wave function of the  $SU(N)$  electrons obtained from the parton construction is still given by eq. (21). But now  $\chi_s$  is the wave function of the first Landau level filled by the partons with  $N$  flavors, which has a filling fraction  $N$ . The total filling fraction of the  $SU(N)$ -electron state (21) is  $\nu = N/k$ . Again using the calculations in Ref. 6, we can show that the effective theory of the state (21) is given by the  $SU(k)$  level  $N$  CS theory and quasiparticles have the non-abelian statistics.

Let us state the above result more accurately. We know the ground state (21) is invariant under the  $SU(N)$  transformations. Thus the quasiparticle excitations carry definite  $SU(N)$  quantum number. Using the effective theory we can show that the ground state (21) supports quasiparticles in the fundamental representation of  $SU(N)$  that carry electric charge  $e/k$ . The statistics of such quasiparticles is that of the Wilson lines in the  $SU(k)$  level  $N$  C-S theory plus an abelian factor. The Wilson lines are in the fundamental representation of  $SU(k)$  and the abelian factor simply describes the fermionic statistics.

The parton construction discussed in this section is a special case of the more general construction discussed in the last section. Using eqs. (11) and (12) we can show that the

wave function (21) is identical to  $\Psi^1$  defined in eq. (13) (or (14)) if we choose  $p = 0$ . An advantage of the parton construction is that one can easily obtain the low energy effective theory using the method discussed in Ref. 6. If the effective theory is correct, we may conclude that the quasiparticles in the state  $\Psi^1$  have the non-abelian statistics described by the  $SU(k)$  level  $N$  C-S theory. This result provides important clue in understanding the microscopic calculation of the non-abelian statistics (see section 3).

## 2.c The Hamiltonian and the Ground State Degeneracy.

In the above we have discussed the wave functions of the NAF states. Now we would like to discuss possible local Hamiltonians for which the electron wave functions proposed above are their exact ground states. For convenience we will concentrate on  $SU(2)$  electrons that carry “spin”- $j$ . In order for the spin- $j$  operator to be the simple current, we need to choose the level of the  $SU(2)$  WZW model to be  $k = 2j$ .

First let us discuss the order of zeros in the wave function  $\Psi$  defined in (14) (or (21)) as two electrons approach to each other. Let us concentrate on the two electrons labeled by  $z_1$  and  $z_2$ . It is convenient to use the the total spin to label the spin states of the two electrons:

$$\Psi(z_1, z_2; J, M; \dots) = \sum_{m_1, m_2} C_{m_1 m_2}^{JM} \Psi(z_1, m_1, z_2, m_2, \dots) \quad (23)$$

where  $C_{m_1 m_2}^{JM} = \langle JM | m_1 m_2 \rangle$  are the C-G coefficients for the spin summation and  $(\dots)$  represents the coordinates and the spins of rest of the electrons. When  $z_1 \rightarrow z_2$  we have

$$\Psi(z_1, z_2; J, M; \dots) \rightarrow (z_1 - z_2)^{\tilde{\eta}(J, M, m_3, \dots, m_n)} \quad (24)$$

Here the exponent  $\tilde{\eta}$  is an integer due to the single-value property of the wave function. Define  $\eta(J, M)$  to be the minimum of  $\tilde{\eta}(J, M, m_3, \dots, m_n)$  for all values of  $m_3, \dots, m_n$ . From the spin rotation invariance we find that  $\eta(J, M) = \eta(J)$  is independent of  $M$

The simplest way to calculate the order of the zeros is to use the operator product expansion of the simple current

$$V_{m_1}^{j=k/2}(z_1) V_{m_2}^{j=k/2}(z_2) \propto \frac{1}{(z_1 - z_2)^{2h}} [C_{m_1 m_2}^{00} + O(z_1 - z_2)] \quad (25)$$

where  $h = k/4$  is the conformal dimension of the simple current  $V^{j=k/2}$ . The eq. (25) implies that the leading order pole in conformal block  $F^j$  appear only in the  $J = 0$  channel. The order of the pole is  $k/2$ . After including the  $U(1)$  factors in (14) we find that

$$\eta(0) = p, \quad \eta(J \neq 0) > p \quad (26)$$

This result indicates that the order of the zeros has a non-trivial dependence on the total spin of the two electrons.

More detailed information about the zeros can be obtained from the parafermion or the parton constructions of the wave function. From the parafermion construction (see Appendix B) we find the zeros of the conformal block  $F^j$  are given by

$$F^j(z_i, m_i)|_{z_1 \rightarrow z_2} \rightarrow (z_1 - z_2)^{\gamma(m_1, m_2)} \quad (27)$$

$$\gamma(m_1, m_2) = -\frac{k}{2} + |m_1 + m_2|$$

Using the spin rotation symmetry we find that the above result implies

$$\eta(J) = p + J \quad (28)$$

Once we know the orders of zeros in the wave function as  $z_i \rightarrow z_j$ , it is easy to write down local Hamiltonians for which the wave function is an exact ground state. In our case let us consider the following Hamiltonian

$$H = \sum_i \left[ \frac{1}{2m} (\partial - iA)^2 - \frac{1}{2} \omega_c \right] + \sum_{i < j, J} (-)^{\eta(J)-1} C_J P_J(S_i, S_j) \Delta_{\eta(J)}(r_i - r_j) \quad (29)$$

$$\Delta_n(r) = (1 - \delta_{n,0}) \partial_r^{n-1} \delta^{(2)}(r) \partial_r^{n-1}$$

Here  $\omega_c$  is the cyclotron frequency,  $P_J(S_i, S_j)$  is the projection operator into  $(S_i + S_j)^2 = J(J+1)$  channel and  $S_i$  is the spin operator of the electrons. It is easy to show that the Hamiltonian is positive definite if all  $C_J \geq 0$ . It is also obvious that the Hamiltonian has zero expectation value on our wave function. Therefore the wave function we considered above is an exact ground state of the Hamiltonian (29).

Two questions remain. A) What is the degeneracy of the ground states? B) Is the ground state incompressible, *i.e.*, does the gap of the excitations remain finite in the thermodynamical limit? In order to answer on these two questions we need to put our system on a compactified Riemann surface in order to avoid gapless edge excitations (which always exist). The answer to the question A) is that the ground state degeneracy is equal to the number of the conformal blocks on the Riemann surface. This is because the different conformal blocks have the same orders of zeros as the electron coordinates  $z_i \rightarrow z_j$ . Thus they all have the same zero energy. According to the conformal theory, the number of the conformal blocks, hence the ground states degeneracy, depends on the topology of the Riemann surface. Such topology-dependent degeneracies characterize the non-trivial internal correlations ( the topological orders) in our NAF states. Measuring the ground states degeneracy on the Riemann surfaces is the simplest way to measure the internal structures of the FQH states in numerical calculations.<sup>31</sup>

The question B) is much more difficult to answer. Even the proof of the incompressibility of the Laughlin states is still missing, except some numerical evidence from calculations in finite systems. However from the experience of many numerical calculations, we expect that our NAF states are incompressible for the Hamiltonian (29) with a generic choice of  $C_J$ . This is because the Hamiltonian select a particular pattern of zeros that characterizes the wave function.

We would like to remark that for more general Hamiltonians (say with an weak random potential added to (29)), the ground states may not be exactly degenerated for the finite systems. However the energy splits between different ground states are expected to be exponentially small as the size of the system goes to infinity (as it happens, for example, in the abelian system on the torus Ref. 31).

### 3. QUASIPARTICLES AND NON-ABELIAN STATISTICS

In the last section we discussed the ground state wave functions and the Hamiltonians for which these wave functions are the exact ground states. In this section we would like to discuss the quasiparticles and their statistics.

We know that the ground state wave function can be expressed as correlation function of simple currents in the K-M algebra. We would like to propose that the quasiparticle excitations are created by the insertions of the other types of the primary fields (not necessarily simple currents) of the K-M algebra. The explicit electron wave function in the presence of the quasiparticles is given by eq. (17).

Let us ask the following physical question: why eq. (17) represents local quasiparticle excitations with finite energies? First, notice that the orders of zeros as  $z_i \rightarrow z_j$  remain unchanged after insertion of the quasiparticles. Therefore eq. (17) is a zero energy eigenstate of the Hamiltonian (29). It is less trivial to show that the expression (17) represents *local* excitations. Here by local excitations we mean that all physical quantities (such as densities, density correlations *etc.*) approach to their “vacuum” values if measured far away from the excitations. In the following we will give a heuristic argument about the locality of the quasiparticles. First let us ask the same question for the quasiparticles in the Laughlin state which have a wave function:

$$\begin{aligned}\Phi &= \prod_k (z_k - \eta) e^{-\frac{1}{\sqrt{m}}|\eta|^2} \Phi_m \\ \Phi_m &= \prod_{i < j} (z_i - z_j)^m \prod_k e^{-\sqrt{m}|z_k|^2}\end{aligned}\tag{30}$$

Here  $m$  is an odd integer. We know that the quasiparticle at the point  $\eta$  behaves like a flux tube (of unit flux). Mathematically this means that the wave function satisfies the following K-Z type equation:

$$\begin{aligned}(\nabla_{z_{i_0}} - \frac{1}{z_{i_0} - \eta} - \sum_{i \neq i_0} \frac{m}{z_{i_0} - z_i})\Phi &= 0 \\ \nabla_{\bar{z}_{i_0}}\Phi &= 0\end{aligned}\tag{31}$$

where  $\nabla_z = \partial_z + q\bar{z}$  and  $\nabla_{\bar{z}} = \partial_{\bar{z}} + qz$  are the covariant derivatives in the presence of the constant magnetic field. For the above Laughlin state,  $q = \sqrt{m}$ . We have chosen the magnetic length  $l_B$  to be  $2/q$ . (Note that each electron also behaves like a flux tube (of  $m$  units of flux). Since the gauge field of a flux tube is trivial if we measure it far away from the center, the physical correlations will remain the same as those in the ground state. This can be thought as the reason why the Laughlin quasiparticle is a local excitation. In terms of the more traditional language, the K-Z equation simply means that the quasiparticles are created by turning on the unit flux.

The wave function of the NAF state with the quasiparticles (see (17)) also satisfies the K-Z equation

$$\begin{aligned}&\left( \nabla_{z_{i_0}} - \frac{1}{k + c_V} \sum_I \frac{T_{a_I, a'_I}^a T_{\lambda_{i_0}, \lambda'_{i_0}}^a}{z_{i_0} - \eta_I} - \frac{1}{k + c_V} \sum_{\lambda'_i, i \neq i_0} \frac{T_{\lambda_i, \lambda'_i}^a T_{\lambda_{i_0}, \lambda'_{i_0}}^a}{z_{i_0} - z_i} \right) \Psi(z_i, \lambda'_i; \eta_I, a'_I) \\ &- \left( \sum_I \frac{qq_I}{z_{i_0} - \eta_I} + \sum_{\lambda'_i, i \neq i_0} \frac{q^2}{z_{i_0} - z_i} \right) \Psi(z_i, \lambda'_i; \eta_I, a'_I) = 0\end{aligned}\tag{32}$$

and  $\nabla_{\bar{z}_{i_0}} \Psi = 0$ . Again we see that the quasiparticle in the NAF state correspond to a flux tube, but now the flux tube carries a non-abelian flux. Away from the center of the flux tube the gauge field is trivial and should not change the correlation in the wave function. Thus we expect that the flux tubes (*i.e.*, the quasiparticles) correspond to the local excitations. According to the above picture we see that the quasiparticles in the NAF states can be created by turning on the non-abelian flux.

Now let us calculate the statistics of the quasiparticles in the expression (17). In order for the statistics to be well defined, we need to assume that the ground state wave function given by eq. (14) represents an incompressible state. First we notice that the electron wave function in the presence of the quasiparticles is not unique even for fixed  $\eta_I$  and  $a_I$ . The different electron wave functions correspond to the different conformal blocks labeled by  $\alpha$ . However for the Hamiltonian (29), all the conformal blocks are degenerate (and have zero energy) because the orders of the zeros for all different blocks are the same as those in the ground state.

We know that the statistics of the quasiparticles is defined as the Berry's phase arising from interchanging two identical quasiparticles. Because of the degeneracy between conformal blocks, the Berry's phase may be non-abelian (*i.e.*, it can contain mixing between different blocks). For convenience let us fix  $\Lambda_I$  and  $a_I$  for all quasiparticles, and drop these indices in the following discussion. Mathematically the non-abelian Berry's phase is a matrix given by

$$U = MP \exp(i \int_0^1 dt A) \quad (33)$$

where  $P$  represents the path ordered product,  $A$  and  $M$  are the matrices defined by

$$\begin{aligned} A_\beta^\alpha(t) |\Phi(\eta(t), \beta)\rangle &= i \hat{P} \frac{d}{dt} |\Phi(\eta(t), \alpha)\rangle \\ M_\beta^\alpha |\Phi(\eta(t=0), \beta)\rangle &= |\Phi(\eta(t=1), \alpha)\rangle \end{aligned} \quad (34)$$

In eq. (34),  $\hat{P}$  is the projection into the ground state that is given by

$$\hat{P} = \sum_{\bar{\gamma}\lambda} g_{\lambda\bar{\gamma}} |\Phi(\eta(t), \bar{\gamma})\rangle \langle \Phi(\eta(t), \lambda)| \quad (35)$$

$|\Phi(\eta(t), \alpha)\rangle$  is the electron wave function,  $\alpha, \beta$  label the conformal blocks and  $\eta(t)|_{t \in [0,1]}$  is the path through which the quasiparticles are exchanged. Here  $g_{\lambda\bar{\gamma}}$  is the inverse of  $g^{\bar{\gamma}\lambda}$  given by

$$g^{\bar{\gamma}\lambda} = \langle \Phi(\eta(t), \gamma) | \Phi(\eta(t), \lambda) \rangle \quad (36)$$

We have omitted the  $SU(N)$  indices of the quasiparticles in eq. (34). Note that eq. (33) is valid even when the wave functions labeled by different  $\alpha$ ,  $|\Phi(\eta(t), \alpha)\rangle$ , are not orthogonal to each other.

When the electron wave functions are given by the conformal blocks as in eq. (17), the matrix  $M$  in eq. (34) is just the monodromy matrix in the conformal field theory. However there is one point that must be clarified. The matrix  $M$  in eq. (34) is defined through a fixed exchange path. The number of the electrons enclosed in the exchange path is not fixed. While the monodromy matrix in the conformal field theory is defined with a fixed number of primary fields inside the exchange path. In general, the monodromy matrices are different if the exchange path encloses different primary fields. However in

our case, since the monodromy matrix between electrons and the quasiparticle is trivial, the monodromy matrix that arises from the exchange of two quasiparticles is independent from the number of the electrons enclosed in the exchange path. Therefore the matrix  $M$  in eq. (34) is well defined and coincides with the monodromy matrix in the conformal field theory.

However from eq. (33) we see that the non-abelian statistics (*i.e.*, the non-abelian Berry's phase) is not only determined by the monodromy matrix but is also dependent on the connection  $A$ . The potential effects of this connection are overlooked in Ref. 14. In the following we will concentrate on the calculation of the connection  $A$ .

Let us introduce

$$\begin{aligned} (A_{\eta_I})_{\beta}^{\alpha} &= \sum_{\bar{\gamma}} ig_{\beta\bar{\gamma}} \langle \Phi(\eta, \bar{\gamma}) | \frac{\partial}{\partial \eta_I} | \Phi(\eta, \alpha) \rangle \\ (A_{\bar{\eta}_I})_{\beta}^{\alpha} &= \sum_{\bar{\gamma}} ig_{\beta\bar{\gamma}} \langle \Phi(\eta, \bar{\gamma}) | \frac{\partial}{\partial \bar{\eta}_I} | \Phi(\eta, \alpha) \rangle \end{aligned} \quad (37)$$

Then the connection  $A$  is given by

$$A = A_{\eta_I} \frac{d\eta_I}{dt} + A_{\bar{\eta}_I} \frac{d\bar{\eta}_I}{dt} \quad (38)$$

Because the wave function (17) is holomorphic in  $\eta_I$  except the Gaussian term  $e^{-q_I|\eta_I|^2}$ , the eq. (37) can be written as

$$\begin{aligned} (A_{\eta_I})_{\beta}^{\alpha} &= -iq_I \bar{\eta}_I \delta_{\beta}^{\alpha} + (\delta A_{\eta_I})_{\beta}^{\alpha} \\ (\delta A_{\eta_I})_{\beta}^{\alpha} &= \sum_{\bar{\gamma}} ig_{\beta\bar{\gamma}} \frac{\partial}{\partial \eta_I} g^{\bar{\gamma}\alpha} \\ (A_{\bar{\eta}_I})_{\beta}^{\alpha} &= iq_I \eta_I \delta_{\beta}^{\alpha} + (\delta A_{\bar{\eta}_I})_{\beta}^{\alpha} \\ (\delta A_{\bar{\eta}_I})_{\beta}^{\alpha} &= - \sum_{\bar{\gamma}} ig_{\beta\bar{\gamma}} \frac{\partial}{\partial \bar{\eta}_I} g^{\bar{\gamma}\alpha} \end{aligned} \quad (39)$$

Eq. (39) reduces the calculation of the connection  $A$  to a calculation of the inner product  $g^{\bar{\alpha}\beta}$ .

To gain some intuition about the properties of the inner product  $g^{\bar{\alpha}\beta}$ , let us again consider the Laughlin state. The one quasiparticle state is described by the wave function (30) whose norm can be shown to be independent of the position of the quasiparticle  $\eta$  (note that the Gaussian factor  $e^{-\frac{1}{\sqrt{m}}|\eta|^2}$  is important for this to be true). Despite the proper normalization of the one-quasiparticle state, the two-quasiparticle state given by

$$\tilde{\Phi}(\eta_1, \eta_2) = \prod_k (z_k - \eta_1)(z_k - \eta_2) e^{-\frac{1}{\sqrt{m}}(|\eta_1|^2 + |\eta_2|^2)} \Phi_m \quad (40)$$

is *not* properly normalized. In the fact one can show that

$$\langle \tilde{\Phi}(\eta_1, \eta_2) | \tilde{\Phi}(\eta_1, \eta_2) \rangle \propto |\eta_1 - \eta_2|^{-2/m} \quad \text{for } |\eta_1 - \eta_2| \gg l_B \quad (41)$$

However if we use the correlation function of  $U(1)$  WZW model to express the two-quasiparticle state, we find that the wave has a form

$$\Phi(\eta_1, \eta_2) = (\eta_1 - \eta_2)^{1/m} \prod_k (z_k - \eta_1)(z_k - \eta_2) e^{-\frac{1}{\sqrt{m}}(|\eta_1|^2 + |\eta_2|^2)} \Phi_m \quad (42)$$

which contains an additional factor  $(\eta_1 - \eta_2)^{1/m}$  respective to the r.h.s. of eq. (40). Due to this factor, the wave function given by the  $U(1)$  model is properly normalized, *i.e.*, the norm of the state (42) is independent from the quasiparticle positions  $\eta_1$  and  $\eta_2$  when the two quasiparticle are well separated. This implies that the connection  $\delta A = 0$  since  $g^{\bar{\alpha}\alpha}$  is independent of  $\eta_I$ . In this case the connection  $A = -\frac{i}{\sqrt{m}} \sum_I (\bar{\eta}_I \frac{d\eta_I}{dt} - \eta_I \frac{d\bar{\eta}_I}{dt})$  describes the interaction of the quasiparticle charge with the background magnetic field. It is a property of the single particle and is independent of the relative positions between different particles. Thus the statistics of the quasiparticle is given by the monodromy matrix  $M$ . From the eq. (42) we see that the wave function is not single valued when considered as a function of  $\eta_I$  (note that the wave function (42) is always single valued when it is considered as a function of the electron coordinates) and the half-monodromy matrix is non-trivial:  $M = e^{i\pi/m}$ . Thus the statistics of the quasiparticle is given by  $\theta = \pi/m$ .

The above results can be stated as follows. If the two-quasiparticle wave function is chosen to satisfy the K-Z equation for both the electron coordinates  $z_i$  and the quasiparticle coordinates  $\eta_I$ , then the inner product of the wave function,  $\langle \Phi(\eta_I) | \Phi(\eta_I) \rangle$ , is *independent* of the quasiparticle positions once the quasiparticles are well separated (compared to the magnetic length  $l_B$ ). For such choice of wave the function, the connection  $A$  does not contribute to the statistics and the quasiparticle statistics is determined only by the monodromy matrix of the wave function.

The structure of the wave function of the NAF state is similar to the structure of the wave function of the Laughlin state. In particular the wave function of the NAF state also satisfies the K-Z type equation, but now the electron and the quasiparticle carry the non-abelian flux. It is natural to speculate that the above results for the abelian case also apply to the non-abelian case. For example the inner product  $g^{\bar{\alpha}\beta}$  may be independent of the quasiparticle positions if the wave function is chosen to satisfy the K-Z equation. If this speculation is correct, the quasiparticle statistics will be given by the monodromy matrices of the wave function. Furthermore the connection will have a form  $A = -i \sum_I q_I (\bar{\eta}_I \frac{d\eta_I}{dt} - \eta_I \frac{d\bar{\eta}_I}{dt})$  that describes solely the interaction of the quasiparticle charge with the background magnetic field. This implies that the quasiparticle charge  $Q$  is given by  $q_I/q$ . For the quasiparticles in the fundamental representation we have

$$Q = \nu \frac{l}{N} = \frac{l}{kl^2 + pN} \quad (43)$$

When  $p = 0$  and  $l = 1$  the above result reduces to the one obtained in Ref. 6.

To test the above speculation, let us consider the wave functions which can be obtained by parton construction (see (21)). Those wave functions describe the  $SU(N)$  electrons in  $k\Lambda^1$  representation. (For simplicity we also assume  $p = 0$ ) For those states the quasiparticle statistics can be calculated independently from the effective C-S theory. The quasiparticles were shown to have a non-abelian statistics described by  $SU(k)$  level  $N$  C-S theory.

In the following we would like to calculate the non-abelian Berry's phase for the quasiparticles in the state (21) assuming the connection  $\delta A = 0$  and comparing the results with

those obtained from the effective theory. When  $\delta A = 0$ , the non-abelian statistics of the quasiparticles is given by the braiding matrix of the conformal blocks.

Let us consider a case when there are four quasiparticles; two in  $\Lambda^1$  representation and the other two in the conjugate representation  $\bar{\Lambda}^1$ . The two quasiparticles in  $\Lambda^1$  representation are created by the operator  $\exp(i\bar{q}_1\phi)V^{\Lambda^1}$  while the two in  $\bar{\Lambda}^1$  representation are created by the operator  $\exp(iq_1\phi)V^{\Lambda^1}$  where  $q_1 = 1/\sqrt{Nk}$ . We will put our system on a sphere to avoid the edge excitations. In this case the quasiparticle state contains only two conformal blocks. Using the operator product expansion we can find the eigenvalues of the braiding matrix associated with the exchange of the two identical quasiparticles in the  $\bar{\Lambda}^1$  representation

$$B = \exp(i\pi/Nk)B(N, k) \quad (44)$$

Here  $B(N, k)$  is the braiding matrix in the  $SU(N)_k$  WZW model

$$B(N, k) = \begin{pmatrix} \exp(-i\pi(1 - N)/N(N + k)) & 0 \\ 0 & -\exp(-i\pi(1 + N)/N(N + k)) \end{pmatrix} \quad (45)$$

and  $\exp(i\pi/Nk)$  is the contribution from the  $U(1)$  model. Now let us calculate the braiding matrix using the effective theory. According to Ref. 6 the effective theory of the states (21) obtained by the parton construction is given by the  $SU(k)_N$  C-S theory. A careful calculation indicates that the C-S term contains an addition minus sign compared to the standard definition of the C-S theory. As a consequence the braiding matrix given by the effective C-S theory has a form

$$B = -B^\dagger(k, N) \quad (46)$$

Here the minus sign comes from the fermionic statistics of the parton. The braiding matrix (46) is the same for the two particles in the  $\Lambda^1$  representation or in  $\bar{\Lambda}^1$  representation due to the charge conjugation symmetry. Using eq. (45) it is easy to check that the braiding matrices in eqs. (44) and (46) are identical.

The relation  $B(N, k)B(k, N) = -\exp(-i\pi/Nk)$  is one of the consequences of the level-rank duality in the theory of the Kac-Moody algebras (see e.g. Ref. 32 for the recent review). The similar relations hold for fields in other representations. We expect that all braiding matrices calculated from the effective C-S theory and from the microscopic wave function (by assuming  $\delta A = 0$ ) are identical to each other.

The above comparison leads us to the following conclusion. For the states obtained from the parton construction, the metric  $g^{\bar{\alpha}\beta}$  is independent of quasiparticle positions and the connection is given by  $A = -i \sum_I q_I (\bar{\eta}_I \frac{d\eta_I}{dt} - \eta_I \frac{d\bar{\eta}_I}{dt})$ . We expect that the same is true for the other wave functions constructed in Section 2 because these states are very similar to each other. Thus we believe that the statistics of the quasiparticles is given by the braiding matrices for the states constructed in this paper.

To gain a physical understanding of the above result, we notice that the connection

$\delta A$  can be calculated from

$$\begin{aligned}
& g^{\bar{\alpha}\gamma}(\delta A_{\eta_{I_0}})_{\gamma}^{\alpha} \\
& = \langle \Psi(\eta_I, a_I, \alpha) | \left( \frac{1}{k + c_V} \sum_{a'_I, I \neq I_0} \frac{T_{a_I, a'_I}^a T_{a_{I_0}, a'_{I_0}}^a}{\eta_{I_0} - \eta_I} + \frac{1}{k + c_V} \sum_{\lambda'_i, a_{I_0}} \frac{T_i^a T_{a_{I_0}, a'_{I_0}}^a}{\eta_{I_0} - z_i} \right) | \Psi(\eta_I, a'_I, \alpha) \rangle \\
& - \langle \Psi(\eta_I, a_I, \alpha) | \left( \sum_{I \neq I_0} \frac{q_{I_0} q_I}{\eta_{I_0} - \eta_I} + \sum_i \frac{q q_{I_0}}{\eta_{I_0} - z_i} \right) | \Psi(\eta_I, a'_I, \alpha) \rangle
\end{aligned} \tag{47}$$

where we have used the K-Z equation.  $T_i^a$  in eq. (47) are the operators acting on the  $i^{\text{th}}$  electron whose matrix elements are given by  $T_{\lambda_i, \lambda'_i}^a$ . Let first concentrate on the abelian part, the second term in (47). Suppose that the wave function has a plasma analogue. The vanishing of the second term can be understood from the charge neutrality of the wave function (*i.e.*, the screening of the plasma). The term  $\frac{q_{I_0} q_I}{\eta_{I_0} - \eta_I}$  represents the interaction of the quasiparticle at  $\eta_{I_0}$  with the one at  $\eta_I$ . The quasiparticle at  $\eta_I$  repels the electron and creates a vacancy in the electron fluid. Thus the sum  $\sum_i \frac{q q_{I_0}}{\eta_{I_0} - z_i}$  contains a term that describes the interaction of the quasiparticle at  $\eta_{I_0}$  with the vacancy. Such a term has a form  $\frac{q_{I_0} \bar{q}_I}{\eta_{I_0} - \eta_I}$  where  $\bar{q}_I$  is the ‘‘charge’’ of the vacancy. The charge neutrality condition implies that  $q_I + \bar{q}_I = 0$  (we assume that the quasiparticles are well separated). Thus the abelian part of  $\delta A$  is zero. (More precisely  $\delta A = O(\frac{1}{(\eta_{I_0} - \eta_I)^2})$  due to possible dipole interaction. Such term will not contribute to the quasiparticle statistics).

Similarly the vanishing of the non-abelian part of  $A$  (the first term in (47)) can be related to the local singlet property of the wave function. Suppose the state  $|\Psi(\eta_I, a'_I, \alpha)\rangle$  is locally singlet, *i.e.*,

$$\left( \sum_{i, \lambda'_i} f(z_i) T_{\lambda_i, \lambda'_i}^a + \sum_{I, a'} f(\eta_I) T_{a_I, a'_I}^a \right) \Psi(z_i, \lambda'_i, \eta_I, a'_I; \alpha) = 0 \tag{48}$$

where  $f(z)$  is a smooth function (at the scale  $l_B$ ) which is non-zero only in a local region. ((48) with  $f = 1$  is the statement about the global singlet.) Then the non-abelian charge carried by the quasiparticle is canceled by the non-abelian charge of the vacancy created by the quasiparticle. Thus we expect that the first term in eq. (47) also vanishes.

Certainly the plasma analogue and the local singlet property of the non-abelian FQH states have not been established yet. The above discussion is highly speculative. However we believe that the main physical picture and the results presented above are correct.

#### 4. UNIVERSALITY OF THE NON-ABELIAN STATISTICS

In the last two sections we have constructed many-body systems with the non-abelian statistics. We showed that the particular Hamiltonian (29) has the following two properties. A)

The low energy states in the presence of the quasiparticles are degenerated even when the positions and the  $SU(N)$  quantum numbers of the quasiparticles are fixed. The degeneracy is due to the existence of the different conformal blocks. B) The non-abelian Berry's phase associated with exchanging quasiparticles depends only on the topology of the path (except for the overall abelian phase due to the external magnetic field which does depend on the area enclosed by the path). A) and B) are the basic requirements in order for the non-abelian statistics to be well defined.

An important physical question is whether the non-abelian statistics is robust against arbitrary perturbations or not. In particular we would like to know whether the non-abelian statistics discussed in this paper remains to be valid in the absence of the exact  $SU(N)$  symmetry as what happens in experiments.

Unfortunately we can not provide a rigorous answer to the above problem. So in the following we will merely state our opinion on this problem.

We believe that the non-abelian statistics, just like the abelian statistics, is robust against any perturbations. This implies that the degeneracy of different conformal blocks cannot be lifted by *any* perturbations when the quasiparticles are well separated. The degeneracy that we encountered here is very similar to the degeneracy of the FQH states on the Riemann surfaces with the genus  $g > 0$ . Both degeneracies are protected by the topological orders and are robust against the arbitrary perturbations.<sup>31</sup>

The above conjecture has some interesting consequences. We know that the perturbation of the Hamiltonian may couple to the electron density,  $SU(N)$  charge density, and/or their correlations. For example in the presence of the random potential  $V(x)$  the perturbation of the Hamiltonian has the form:

$$\delta H = \int d^2x V(x)\rho(x) \quad (49)$$

where  $\rho$  is the electron density. According to the first order perturbation theory, the robustness of the degeneracy implies that the electron density, the  $SU(N)$  charge densities, and any other local quantities must be the same for all the different conformal blocks. (Certainly the quasiparticles are always assumed to be well separated.)

It is very interesting to check the above results by doing some direct calculations. If our conjecture is correct, the states constructed in this paper represent the new universality classes. The non-abelian statistics may appear for a large class of Hamiltonians. Under right condition those new states may appear in experiments.

In Appendix C we attempt to develop a plasma analogue for the non-abelian FQH states using Feigin-Fuchs representation of the wave functions. The partition function we obtained resemble a two component coulombic plasma, but it also contains some non-coulombic terms. The role of the non-coulombic term is not clear at the moment. However if we are only concerned with the calculation of the filling fraction, it can be argued that the non-coulombic term is not important. We can show that the two component plasma will reproduce the filling fraction obtained in Section 2.A.

## 5. CONCLUSION.

In this paper we discussed the microscopic theory of the non-abelian FQH states. We constructed the many-body systems whose long range properties are described by the non-abelian C-S theory. The elementary excitations in these systems were shown to have the

non-abelian statistics. We believe that the fractional statistics is a topological property of the systems which does not depend on “fine tuning” and short distant details of the systems.

Our results have two applications. First, robustness of the non-abelian FQH states makes it possible to realize those new states experimentally, especially in the low density limit and/or in systems which contain several layers of the 2D electron gas. Second, our systems may be useful as the alternative way to study non-abelian C-S theories.

Let us summarize in detail the results obtained in this paper, in particular we will point out where the speculative arguments were used in order to reach the results. We first used conformal blocks in WZW model to construct wave functions of  $SU(N)$  electrons. The wave functions were shown to be single valued, antisymmetric and well defined in the thermodynamic limit, if the electrons are in a representation characterized by the highest weight  $k\Lambda^l$ . We then constructed a class of Hamiltonians for which the above wave functions were shown to be exact ground states of those Hamiltonians. However we did not show that our wave functions represent an incompressible state. But from previous numerical experience with the FQH states, we speculate that the states that we constructed are incompressible.

Given an incompressible state, one can naturally ask what is the statistics of the quasiparticle excitations. We found that there are two independent ways to calculate the statistics. One way is to use effective C-S theory which, at the moment, is only known for the states which can also be obtained through the parton construction (which require the electrons to be in the representation of highest weight  $k\Lambda^1$ ). These states support the non-abelian statistics. The other way is to calculate the non-abelian Berry’s phase directly from the microscopic wave functions. Unfortunately the second calculation has not been carried out to the end. We failed to give a rigorous calculation of the connection term  $A$  in the Berry’s phase, because we do not have the plasma analogue or something similar for the non-abelian FQH states. However many facts suggest that the connection  $A$  is given by

$$A = -i \sum_I q_I (\bar{\eta}_I \frac{d\eta_I}{dt} - \eta_I \frac{d\bar{\eta}_I}{dt}). \quad (50)$$

when the quasiparticles are well separated. This result is supported by the analogue with the Laughlin wave function. For the electrons in the  $k\Lambda^1$  representation, (50) can be derived from the effective theory. We also discussed how to understand eq. (50) from the local singlet property of our states. This makes us to believe that eq. (50) is also valid for the states which cannot be obtained from the parton construction.

It is important to know whether the non-abelian FQH states constructed in this paper are generic states or not. Our opinion (which again has not been proven rigorously) is that the non-abelian statistics is a topological quantity which cannot be changed by small perturbations of any form (including those that break the  $SU(N)$  symmetry). It can only be changed through phase transitions. Thus we believe that our states represent new universality classes of strongly correlated electron systems. As a consequence the degeneracy associated with the non-abelian statistics is expected to be robust against any perturbations. We believe that the degeneracy is protected by the non-trivial topological order in our states, like the degeneracy of the FQH states on torus. This result implies that all the local physical quantities, such as densities and their correlations, must be the same for all the different degenerated states.

We would like to remark that the effective theory of the non-abelian FQH states does depend on perturbations of the Hamiltonian. However the perturbation only contribute

to the higher order derivative terms, such as the Maxwell term. These changes are related to the change in the short distance structures of the states. The C-S term can never be changed by small perturbations due to the gauge invariance. This fact is related to the robustness of the long distance structure (or the topological orders) in our states. The difficulties that we encountered in microscopic calculation are related to the problem of how to separate the long distance properties from the short distance ones.

We also calculated many other important properties of the non-abelian FQH states. For example the state given by eq. (14) was shown to have a filling fraction  $\nu = \frac{N}{kl^2 + Np}$ . The quasiparticle created by, say,  $e^{iq_I\phi}V^\Lambda$  was found to have an electric charge  $Q = q_I/q = \frac{l}{kl^2 + pN}$  where  $q$  and  $q_I$  are given by eqs. (9) and (18). It also carries a  $SU(N)$  charge in  $\Lambda^1$  representation. In particular  $Q = e/k$  if the electrons are created by the simple current in the  $k\Lambda^1$  representation and  $p = 0$  in eq. (9).

We believe that many nice properties of the states constructed in this paper are the consequences of the K-Z equation. Physically speaking the K-Z equation describes binding flux to electrons. The non-abelian version of the K-Z equation tells us how to bind the non-abelian flux to the  $SU(N)$  electrons in a consistent way. Because both Laughlin states and the non-abelian FQH states are obtained by binding flux to electrons, we expect that many physical pictures and intuitions developed for the Laughlin states also apply to the non-abelian FQH state. The conformal symmetry appears to play little role in our discussion. It seems that the electron states may have different properties if they are constructed from the conformal blocks which do not satisfy a K-Z type equation.

Certainly many things remain to be done. In particular it would be interesting to find realistic systems whose Hamiltonians resemble the Hamiltonian given by eq. (29) and to observe non-abelian FQH states experimentally.

## APPENDIX A. WZW MODELS AND KAC-MOODY ALGEBRAS

Let us recall some useful facts from the representation theory of the Kac-Moody algebras. These algebras are generated by the conserved currents of the WZW model. They obey the following operator product expansion:<sup>33,23,34</sup>

$$J^a(z)J^b(z') = \frac{k}{(z-z')^2}\delta^{ab} + \frac{f^{abc}}{z-z'}J^c(z') + \dots \quad (51)$$

The corresponding conformal field theory was discussed in Ref. 23,34. The primary fields  $V_\lambda^\Lambda$  are labeled by the weights  $\lambda$  that belong to the highest weight representations  $\Lambda$  of the  $SU(N)$  Lie algebras. The conformal fields are all annihilated by the positive frequency modes of the Kac-Moody currents. The fields with different  $\lambda$  but the same  $\Lambda$  are connected by the action of zero modes  $J_0^a$  of the currents. The primary fields have the operator product expansions with the currents:

$$J^a(z)V_\lambda^\Lambda(z') = \frac{(t^a)_{\lambda\lambda'}V_{\lambda'}^\Lambda}{(z-z')} + \dots \quad (52)$$

Here  $(t^a)_{\lambda\lambda'}$  are the matrix generators of the  $SU(N)$  Lie algebra.

We now recall three different ways to represent the chiral WZW model that are used in this paper. First, there is the representation that uses the chiral fermions<sup>15,16</sup>. Consider the fermions  $\psi_s^i$  that form the fundamental representation of the WZW model. We take  $k$  sets of these fermions, where  $k$  is the level of the representation. The primary fields can be expressed as the contractions of the fermions by the  $SU(k)$  tensors. The construction of the wave functions based on this representation of the WZW model is very similar to the parton construction used in Ref. 6 and in section 2B.

The second construction of the WZW model is based on the Feigin-Fuchs representation. In this approach we represent the model in terms of the free scalar fields coupled to the charge at infinity (the curvature anomaly). We refer the reader to Ref. 17,35,36 for the detailed description of this approach.

Finally, the WZW model can be related to the parafermionic conformal field theories.<sup>18</sup> The vertex operators of these two models are connected. For the case of the  $SU(2)$  Kac-Moody algebra

$$V_m^j = G_m^j \exp(im\phi/\sqrt{k}) \quad (53)$$

Here  $G_m^j$  is the parafermionic vertex operator and  $V_m^j$  is the WZW vertex operator. The field  $\phi$  is the free bosonic field. The currents of the parafermionic and the WZW models are connected as

$$\begin{aligned} \psi_1 &= J^+ \exp(-i\phi/\sqrt{k})/\sqrt{k} \\ \psi_1^+ &\equiv \psi_{k-1} = J^- \exp(+i\phi/\sqrt{k})/\sqrt{k} \end{aligned} \quad (54)$$

The conformal dimensions of the parafermions are given by:

$$\begin{aligned} h_m^l &= h_l + \frac{(l-2m)(l+2m)}{4k}, \quad -l \leq 2m \leq l \\ h_m^l &= h_l + \frac{(2m-l)(2k-l-2m)}{4k}, \quad l \leq 2m \leq 2k-l \end{aligned} \quad (55)$$

Here

$$h_l = \frac{l(k-l)}{2k(k+2)}, \quad l = 2j \quad (56)$$

The important property of the parafermionic models is the identification Ref. 30

$$\begin{aligned} G_m^j &= G_{j+m}^{k-j} \\ G_m^j &= G_{m+k}^j \end{aligned} \quad (57)$$

In particular the parafermionic currents are connected to the fields  $V_m^{k/2}$  of the WZW model as

$$V_m^{k/2} = \psi_{k/2-m} \exp(i\frac{m}{\sqrt{k}}\phi) \quad (58)$$

where  $\psi_m = G_m^0$ . This relation will enable us to express the correlation function of the fields  $V_m^{k/2}$  in the WZW model through the correlation function of the parafermionic (and consequently, due to eq. (54), of the Kac-Moody) currents.

The same construction can be realized for the case of the general  $SU(N)_k$  WZW model. In this case we must use the generalized parafermions proposed in Ref. 37.

APPENDIX B. PARAFERMIONIC  
CONSTRUCTION OF THE WAVE FUNCTIONS.

In this section we shall construct the wave functions using the relation between the parafermions and the WZW model. For simplicity we shall consider only the case of the  $SU(2)$  WZW model. From (53) we find that the correlation functions of the fields in the WZW model and in the parafermionic theory are connected as

$$\langle V_{m_1}^{l_1}(z_1) \dots V_{m_n}^{l_n}(z_n) \rangle = \langle G_{m_1}^{l_1}(z_1) \dots G_{m_n}^{l_n}(z_n) \rangle \prod_{i,j=1}^{i,j=n} (z_i - z_j)^{\frac{2m_i m_j}{k}} \quad (59)$$

Here  $m_i$  are the magnetic quantum numbers. In particular, using the identification (57) we can determine the conformal block that corresponds to the ground state wave function. This is the correlator of the simple currents that can be expressed through the correlator of the parafermionic currents.

$$\langle V_{m_1}^{k/2}(z_1) \dots V_{m_n}^{k/2}(z_n) \rangle = \prod_{i,j=1}^{i,j=n} (z_i - z_j)^{2m_i m_j / k} \langle \psi_{-k+2m_1}^0(z_1) \dots \psi_{-k+2m_n}^0(z_n) \rangle \quad (60)$$

In the case when  $k_i - 2m_i = \pm 1$ , the correlator (60) can be expressed through the correlator of the Kac-Moody currents using eq. (54):

$$\begin{aligned} & \langle V_{k/2-1}^{k/2}(z_1) \dots V_{-k/2+1}^{k/2}(z_n) \rangle \\ &= \prod_{i,j=1}^{i,j=n} (z_i - z_j)^{k/2-2} \prod_{i,j=1}^{i,j=n} (\eta_i - \eta_j)^{k/2-2} \prod_{i,j=1}^{i,j=n} (z_i - \eta_j)^{-k/2+2} \langle \prod_{i=1}^{i=n} J^+(z_i) \prod_{i=1}^{i=n} J^-(\eta_i) \rangle \end{aligned} \quad (61)$$

Here  $z_i$  are the points in which  $k_i - 2m_i = 1$ , and  $\eta_i$  are the points where  $k_i - 2m_i = -1$ . All other correlators of the simple currents can be obtained from the correlator (60) making the limit  $z_i \rightarrow z_j$ . This is the consequence of the fact that the parafermionic currents form an additive current algebra:  $\psi_i \psi_j \propto \psi_{i+j}$ .

The parafermionic representation of the wave functions that we discussed in this subsection is useful if we want to find the leading power of zeros in the electron wave functions. It is clear from eqs. (59) and (60) that the order of the zeros that arises when two electron operators  $V_{m_1}^{k/2}$  and  $V_{m_2}^{k/2}$  are brought together is equal to

$$\gamma(m_1, m_2) = 2m_1 m_2 / k - (\Delta_{k-m_1-m_2} - \Delta_{k/2-m_1} - \Delta_{k/2-m_1}) \quad (62)$$

Here  $\Delta_m = h_m^0$  is the conformal dimension of the parafermionic current  $\psi_m$  given by eq. (55). Using the explicit expressions (55) for the conformal dimensions we immediately obtain the eq. (27) in section 2.C. Finally, note that using the Ward identities we can explicitly express the ground state wave function as the rational function.

#### 4A. APPENDIX C

Let us consider the Feigin-Fuchs construction of the  $SU(N)$ -electron wave functions. This construction is based on the Feigin-Fuchs construction of the conformal blocks of the WZW model<sup>17,38,35</sup>. This representation looks like the non-abelian analogue of the coulombic plasma in the abelian case. For simplicity we shall consider here only the case of the  $SU(2)$  WZW model. The expressions for the general  $SU(N)$  WZW model can be easily obtained using the construction of Ref. 38,35. We omit them here because they are rather cumbersome.

Consider first the ground state wave function. This wave function has the integral representation:

$$\Psi(z_1, \dots, z_n) = \int \prod_{i=1}^{i=k(n-1)/2} dt_i G(t_1, \dots, t_{k(n-1)/2}; z_1, \dots, z_n) \quad (63)$$

Here

$$G(t_1, \dots, t_{k/2(n-1)}; z_1, \dots, z_n) = \prod_{i,j=1}^{i,j=k(n-1)/2} (t_i - t_j)^{\frac{-2}{k+2}} \prod_{i,j=1}^{i=k(n-1)/2, j=n} (t_i - z_j)^{\frac{k}{k+2}} \prod_{i,j=1}^{i,j=n} (z_i - z_j)^{-\frac{k^2}{k+2}} L(t_i; z_j) \quad (64)$$

The function  $L$  is given by

$$L = \sum_{\sigma \in S_{k/2(n-1)}} \frac{1}{(t_1 - z_{\sigma(1)}) \dots (t_{k/2(n-1)} - z_{j_{\sigma(k/2(n-1))}})} \quad (65)$$

The symbol  $S_p$  denotes the set of all transmutations of  $p$  numbers. By the set  $z_i$  we understand the set of points  $z_i$  such that each point is repeated  $k^i$  times;  $\sum k^i = k/2(n-1)$ . The magnetic quantum numbers of the electrons are equal to  $m_i = j_i - k^i/2$ . There is only one conformal block in the ground state wave function. There are  $k/2(n-1)$  contours of integration in eq. (63). One contour is for the each variable  $t_i$ . This number can be easily determined using the results of Ref. 38,35.

The excited state wave functions are obtained by inserting the additional multipliers into eq. (65). The general expression for the conformal block with quasiparticles inserted into the points  $x_1, \dots, x_n$  is

$$G(z_1, \dots, z_n; x_1, \dots, x_m) = \int \dots \int \prod_{i=1}^{i=P} dt_i (z_i - z_j)^{-k^2/2(k+2)} (z_i - x_k)^{-j_k/k+2} (z_i - t_s)^{+k/k+2} (x_i - t_s)^{+2j_n/k+2} (t_i - t_j)^{-2/k+2} F(t_i, z_i, x_i) \quad (66)$$

Here the function  $F$  is given by the expression

$$F = \sum_{\sigma \in S_P} \frac{1}{(t_1 - \chi_{\sigma(1)}) \dots (t_P - \chi_{j_{\sigma(P)}})} \quad (67)$$

Here by the set  $\chi_i$  we understand the set of points  $(z_i, x_i)$  each of whom is repeated  $k^i$  times. The number  $P$  is the number of the integration contours,  $P = \sum k^i = \sum_{i=1}^{i=n-1} j_i - j_n$ . The spins  $j_k$  are the spins of the quasiparticles in the points  $x_k$ . The spin  $j_n$  is the spin of one of the representations that we put at infinity (we use the dual vertex operator for this representation). The points  $z_i$  are the coordinates of the electrons and the points  $x_i$  are the coordinates of the quasiholes. The contours of the integration in the formulae above are chosen according to the Felder's prescription <sup>36</sup>.

We shall use the Feigin-Fuchs representation of the wave functions that was described above for the electron density calculation in the ground state. The corresponding electron density is given by

$$\rho_0 = \frac{\int |F(z_1, \dots, z_n)|^2 d^2 z_2 \dots d^2 z_n}{\int |F(z_1, \dots, z_n)|^2 d^2 z_1 2d^2 z_2 \dots d^2 z_n} \quad (68)$$

Here  $F$  is the electron ground state wave function. It is possible to prove Ref. 39,40 that

$$|F(z_1, \dots, z_n)|^2 = \int \prod d^2 t_i |G|^2 |w|^2 \quad (69)$$

Here  $G$  is the  $U(1)$  factor. The function  $w$  is the integrand of the Feigin-Fuchs representation (64) of the ground state wave function. The expression (69) looks similar to the partition function of the coulombic gas with the particles with the coordinates  $t_i$  and  $z_j$ . There are two components in this coulombic gas. First, there are particles with the coordinates  $t_i$  that correspond to the "internal" component of this gas. These internal particles interact with each other with the charge  $e_1^2 = -\frac{2}{k+2}$ . There are also external particles that interact with the charge  $e_2^2 = \frac{k}{k+2}$ . The particles of the two types interact with each other with the charge  $e_{12}^2 = \frac{k}{k+2}$ . Besides the coulombic type interaction we have also the collective type interaction due to the term  $|L|^2$ .

We now evaluate the integral (68) in the saddle point approximation. We introduce the mean field densities of the  $z$ - and  $t$ - particles. We rewrite the integrand (64) as

$$\begin{aligned} & \int d^2 z d^2 z' g(z) \ln |z - z'| g(z') \left( k/2 - \frac{k^2}{2(k+2)} \right) \\ & + \int |z|^2 g(z) d^2 z + \frac{k}{k+2} \int g(z) \ln |t - z| \rho(t) d^2 z d^2 t \\ & - \frac{2}{k+2} \int d^2 t d^2 t' \rho(t) \ln |t - t'| \rho(t') + \ln(L) \end{aligned} \quad (70)$$

It is easy to see that the contribution of the term  $\ln(L)$  in the integral is of the order  $N$  and is much smaller than the contribution of order  $N^2$  due to the first four terms in the integral (70). We can now use the saddle point approximation in order to evaluate the integral (68) :

$$\left( k/2 - \frac{k^2}{2(k+2)} \right) g + \frac{k}{k+2} \rho = 1 \quad (71)$$

$$\frac{k}{k+2} g - \frac{2\rho}{k+2} = 0 \quad (72)$$

Consequently,

$$g = 2/k \quad \rho = 1 \tag{73}$$

These are the mean field densities for the components of plasma. Both are constants in the saddle point approximation. This result of course coincides with the result of the calculation of the filling fraction obtained in Ref. 6.

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FIGURE CAPTION

**Fig.1a:** Young diagram for the simple current  $k\Lambda_1$ .

**Fig.1b:** Young diagram for the simple current  $k\Lambda_i$ .

**Fig.1:** Young diagrams for the simple currents.