

Chiral Operator Product Algebra
Hidden in Certain Fractional Quantum Hall Wave Functions*

XIAO-GANG WEN

Department of Physics
MIT
77 Massachusetts Avenue
Cambridge, MA 02139

YONG-SHI WU

Department of Physics
University of Utah
Salt Lake City, UT 84112

ABSTRACT: In this paper we study the conditions under which an N-electron wave function for a fractional quantum Hall (FQH) state can be viewed as N-point correlation function in a conformal field theory (CFT). Several concrete examples are presented to illustrate, when these conditions are satisfied, how to “derive” or “uncover” relevant operator algebra in the associated CFT from the FQH wave functions. Besides the known Pfaffian state, the states studied here include three d-wave paired states, one for spinless electrons and two for spin-1/2 electrons (one of them is the Haldane-Rezayi state). It is suggested that the non-abelian topological order hidden in these states can be characterized by their associated chiral operator product algebra, from which one may infer the quantum numbers of quasi-particles and calculate their wave functions.

* Published in *Nucl. Phys.* **B419**, 455 (1994)

1. INTRODUCTION

Fractional quantum Hall (FQH) states contain extremely rich internal structures –as first revealed by the existence of the hierarchical states¹– which one may call topological orders.² While the topological orders in the abelian FQH states are known to be characterized by symmetric integer matrices,³ the characterization of the non-abelian FQH states, i.e. those supporting quasiparticle excitations obeying non-abelian braid statistics, remains an open question. This paper is devoted to the study of the possibility of characterizing some *non-abelian* FQH states by conformal field theory (CFT).

In the literature there have been evidences suggesting that at least some FQH states are closely related to conformal field theories. First, on one hand, the effective theory for abelian FQH states^{4,2,3} and a class of non-abelian FQH states generated by the Kac-Moody algebra⁵ are topological Chern-Simons (CS) theories, which are known to be connected to CFT.⁶ On the other hand, Moore and Read⁷, and also Fubini⁸ independently, have shown that some FQH wave functions can be formally expressed as correlations of primary fields (or vertex operators) in certain CFT. In particular, the correspondence between usual Laughlin wave functions and correlations in the $c = 1$ U(1) CFT's has been recently verified⁹ even for the genus-one case when the FQH state is put on a torus.¹⁰ Furthermore, it has been shown that the quantum numbers of quasiparticles supported by some FQH states are determined by appropriate vertex or disorder operators.^{7,5} Also the ground state degeneracy and the associated non-abelian Berry's phases of FQH states on torus, as shown in Ref. 2,7,9,11, are closely related to conformal blocks and the associated modular transformations in CFT. Especially, the gapless edge excitations of FQH states, by themselves, form chiral CFT.^{12,5,13} All these suggest that at least for some FQH states, the topological order hidden in their wave functions may be characterized by CFT, and the knowledge of the latter is useful in studying topological properties of the FQH state, such as the quantum numbers of quasiparticles, ground state degeneracy and spectrum of edge states. This approach should be helpful particularly for theoretically exploring non-abelian FQH states which are not well understood yet.

It is well-known that when restricted to the first Landau level, apart from an exponential Gaussian factor, a FQH state is described by a holomorphic function $\Phi(z_i)$, where $z_i = x_i + iy_i$ is the complex coordinate of the i -th electron. So the idea has been to use correlation functions in a (chiral) CFT, which are always holomorphic, to generate new non-abelian FQH wave functions in the first Landau level. However, in this approach the starting CFT is chosen somewhat *a priori*; several non-abelian FQH states constructed in this way^{7,5} awaits for their experimental realization. On the other hand, there are FQH states whose many-body wave function are constructed from physical considerations, such as the Haldane-Rezayi wave function for the $\nu = 5/2$ state. Given a FQH state as such, a more natural approach seems to be first to ask whether we can identify the wave function with correlations in a CFT and then, if the answer is yes, to use the operator (product) algebra of the CFT to characterize the topological order hidden in the wave function.

In this paper we will study some conditions under which a given FQH wave function can be identified as correlation functions in a CFT. Then we demonstrate how to derive (the relevant operator algebra in) the associated CFT from the FQH wave function, using the well-understood Pfaffian state as an illustrative example. Using this method, we further construct the operator algebra in the CFT associated to three d-wave paired FQH states respectively, one for spinless electrons and the other two for spin-1/2 electrons (one of them is the Haldane-Rezayi state). The CFT description allows us to study the structure

of quasiparticle excitations in the three d-wave FQH states. The study of other physical consequences, such as the ground state degeneracy on torus and the spectrum of edge states, is in progress and is left to future publication.

We would like to point out that once a FQH wave function is written as correlation function in a certain CFT, then the wave function can also be expressed as a correlation in infinity many other CFT which contain the original CFT. Thus one needs to introduce a concept of minimal CFT in order to define a precise relation between the CFT and FQH wave function. The minimal CFT for a FQH state not only reproduces the wave function, it is also contained in any other CFT that reproduces the wave function. Some times it is not hard to guess a CFT that reproduces a certain wave function. It is often difficult to tell whether the guessed CFT is the minimal one or not. The CFT constructed in this paper is automatically the minimal one. This is because, by construction, the CFT that we obtained contains the minimal set of primary fields that are needed to construct the wave function. Furthermore the energy-momentum tensor of the minimal CFT can be expressed in terms of the operator that generate the wave function. As we will see in a future publication that the minimal CFT are closely related to the edge excitations of the corresponding FQH state.

2. GENERAL DISCUSSION AND THE PFAFFIAN STATE

In general, the holomorphic part of a FQH many-body wave function consists of two factors. One of them is of usual Laughlin-Jastrow form, $\prod_{i<j}(z_i - z_j)^r$ (r is a fraction or integer), which we call the $U(1)$ part. The connection of this part to the $U(1)$ CFT is well-understood. Here we concentrate on the other part, which is not of the Laughlin-Jastrow form. We call it the non-abelian part, because it is this part that is presumably responsible to the appearance of non-abelian statistics for quasiparticles.

We would like to address the following questions: A) Given a holomorphic N -body FQH wave function, is there a CFT whose N -point correlation (for *arbitrary* N) reproduces the FQH wave function. B) If such a CFT exists, how to derive it (or its conformal operator algebra) from the FQH wave function. (For readers who are not familiar with CFT, we recommend the Ref. 14 and the classical paper of Belavin, Polyakov and Zamolochikov.¹⁵)

Mathematically, the affirmative answer to question A) means the following. Viewing the FQH wave function Φ as a correlation of a certain field operator ψ in a 2D quantum field theory,

$$\Phi(z_i) = \langle \prod \psi(z_i) \rangle, \quad z_i = x_i + iy_i \quad (1)$$

we can find an operator $T(z)$ independent of \bar{z} (called the energy-momentum tensor) such that the FQH wave function satisfies the conformal Ward identity¹⁵

$$\langle T(z) \prod_i \psi(z_i) \rangle = \sum_k \left(\frac{h}{(z - z_k)^2} + \frac{1}{z - z_k} \partial_{z_k} \right) \langle \prod_i \psi(z_i) \rangle \quad (2)$$

with a real constant h , called the conformal dimension of the (primary) field ψ . The energy-momentum tensor should also satisfy the Virasoro algebra, which is given, in the form of operator product expansion (OPE), by

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2}{(z_1 - z_2)^2}T(z_2) + \frac{1}{z_1 - z_2}\partial_{z_2}T(z_2) + O(1) \quad (3)$$

The real constant c in (3) is called the central charge. In terms of correlation function, (3) can be written as¹⁵

$$\begin{aligned}
& \langle T(z)T(\tilde{z}) \prod_i \psi(z_i) \rangle \\
&= \sum_k \left(\frac{h}{(z-z_k)^2} + \frac{1}{z-z_k} \partial_{z_k} \right) \langle T(\tilde{z}) \prod_i \psi(z_i) \rangle \\
&+ \left(\frac{2}{(z-\tilde{z})^2} + \frac{1}{z-\tilde{z}} \partial_{\tilde{z}} \right) \langle T(\tilde{z}) \prod_i \psi(z_i) \rangle + \frac{c}{2(z-\tilde{z})^4} \langle \prod_i \psi(z_i) \rangle
\end{aligned} \tag{4}$$

In general the energy-momentum tensor $T(z)$ may or may not be contained in the OPE of the field ψ . If the former is true, this additional information will allow us to express the conditions (2) and (4) as conditions on the wave function Φ . Even in this case, a general solution to the question A) should be quite complicated. So in this paper we restrict ourselves to a simpler question: A') Given a holomorphic FQH wave function, can we write it as the correlations of a primary field ψ in a CFT, which has the following OPE

$$\psi(z_1)\psi(z_2) = \frac{1}{(z_1-z_2)^{2h}} \left(1 + \frac{2h}{c}(z_1-z_2)^2 T(z_2) + O((z_1-z_2)^3) \right) \tag{5}$$

Here the additional simplifying assumption is that the OPE of ψ does not mix with dimension 1 and 2 primary fields.

If the answer to A') is yes, (5) implies that the two-body wave function should have the form $\Phi = 1/(z_1-z_2)^{2h}$. Thus the constant h (the dimension of ψ) can be easily read off from the two-body wave function. (5) also implies that

$$T(z) = \frac{c}{2h} \lim_{z_1 \rightarrow z} \left(\psi(z_1)\psi(z) - \frac{1}{(z_1-z)^{2h}} \right) (z_1-z)^{2h-2} \tag{6}$$

Substituting (6) in to (2) and (4), we find that (2) and (4) become conditions on the wave function Φ , which relate the N -body wave function to the $(N-2)$ - and $(N-4)$ -body wave functions. Thus the question A' can be answered by explicitly checking whether the wave function Φ satisfies these conditions or not.

Let us now apply the above general discussion to a simple example, namely the p-wave paired FQH state for spinless electrons discussed in Ref. 7. The total wave function is given by the product of a Pfaffian wave function Φ_{Pf} and a Laughlin wave function Φ_m

$$\begin{aligned}
\Phi_p &= \Phi_{Pf} \Phi_m \\
\Phi_{Pf} &= \mathcal{A} \left(\frac{1}{(z_1-z_2)} \frac{1}{(z_3-z_4)} \dots \right) \\
\Phi_m &= \left(\prod_{i<j} (z_i-z_j)^m \right) e^{-\frac{1}{4} \sum_i |z_i|^2}
\end{aligned} \tag{7}$$

where \mathcal{A} is the antisymmetrization operator; m is an even integer. The Pfaffian wave function is the exact incompressible ground state of a three-body Hamiltonian.^{16,13,17}

It is known that the Laughlin part of the wave function can be written as correlation of a vertex operator $e^{i\sqrt{m}\phi}$ in the Gaussian model.^{7,8} (Here we adopt the normalization $\langle e^{i\phi(z)} e^{i\phi(0)} \rangle = 1/z$.) We would like to ask whether the Pfaffian part can be written as a correlation of certain operator ψ in an appropriate CFT.

From the two point function $\Phi_{Pf} = 1/(z_1 - z_2)$, we find that ψ has a dimension $h = 1/2$. Φ_{Pf} also has the following property: As $z_1 \rightarrow z_2$

$$\Phi_{Pf}(z_1, \dots) = \frac{\Phi_{Pf}(z_3, \dots)}{z_1 - z_2} + O(z_1 - z_2)$$

Note that no terms are of one power of $z_1 - z_2$ higher than the leading singular term. This implies that (6) defines a valid operator that has well defined correlations with ψ 's. Thus one may try to introduce the energy-momentum tensor through (6), which leads to the identification

$$\langle T(z_2) \prod_{i=3} \psi(z_i) \rangle \equiv c \lim_{z_1 \rightarrow z_2} \left(\Phi_{Pf}(z_1, z_2, \dots) - \frac{1}{z_1 - z_2} \Phi_{Pf}(z_3, z_4, \dots) \right) \frac{1}{z_1 - z_2}. \quad (8)$$

Thus (2) and (4) are reduced to the following conditions on the Pfaffian wave function:

$$\begin{aligned} & c \lim_{z_1 \rightarrow z_2} \left(\Phi_{Pf}(z_1, z_2, \dots) - \frac{1}{z_1 - z_2} \Phi_{Pf}(z_3, z_4, \dots) \right) \frac{1}{z_1 - z_2} \\ &= \sum_{k=3} \left(\frac{1/2}{(z_2 - z_k)^2} + \frac{1}{z_2 - z_k} \partial_{z_k} \right) \Phi_{Pf}(z_3, z_4, \dots) \end{aligned} \quad (9)$$

and

$$\begin{aligned} & c^2 \lim_{z_1 \rightarrow z_2, z_3 \rightarrow z_4} \left(\Phi_{Pf}(z_1, z_2, \dots) - \frac{\Phi_{Pf}(z_3, z_4, \dots)}{z_1 - z_2} - \frac{\Phi_{Pf}(z_1, z_2, z_5, \dots)}{z_3 - z_4} \right. \\ & \quad \left. + \frac{\Phi_{Pf}(z_5, z_6, \dots)}{(z_1 - z_2)(z_3 - z_4)} \right) \frac{1}{(z_1 - z_2)(z_3 - z_4)} \\ &= \sum_{k=5} \left(\frac{1/2}{(z_2 - z_k)^2} + \frac{1}{z_2 - z_k} \partial_{z_k} \right) \langle T(z_4) \prod_{i=5} \psi(z_i) \rangle \\ & \quad + \left(\frac{2}{(z_2 - z_4)^2} + \frac{1}{z_2 - z_4} \partial_{z_4} \right) \langle T(z_4) \prod_{i=5} \psi(z_i) \rangle + \frac{c/2}{(z_2 - z_4)^4} \Phi_{Pf}(z_5, z_6, \dots) \end{aligned} \quad (10)$$

where $\langle T(z_4) \prod_{i=5} \psi(z_i) \rangle$ is given by (8). It is straightforward to verify that the limits in (9) and (10) are indeed finite, and both (9) and (10) are satisfied by the Pfaffian wave function (7) if the central charge c is chosen to be $1/2$. Thus the validity of (9) and (10) assures us that Φ_{Pf} can be identified with correlation of a dimension- $1/2$ field in a $c = 1/2$ CFT. Indeed, the Pfaffian wave function was first constructed⁷ as correlations in the Ising model which has $c = 1/2$.

To give a complete description of a CFT, we also need to know the operator (product) algebra, or OPE, for all primary fields¹⁵. In general the OPE of the ψ operator may generate new operators. Thus ψ can be viewed as a generator which generates a closed

operator algebra, which will be called the center algebra. It is plausible that the properties of this operator algebra (such as structure constants, correlation of new operators generated by ψ , etc.) are determined by the correlations of ψ 's. Then one can use the wave functions to calculate the structure constants of the center algebra. We note that the center algebra is always a semilocal algebra. This is directly related to the single-valueness of the electron wave function. The above semilocal chiral algebra was first discussed in Ref. 7.

For the Pfaffian wave function, since we already know that the wave function coincides with the correlation of the dimension-1/2 field in the Ising model, the knowledge of the latter¹⁵ can be used to determine the operator algebra generated by ψ , which is simply

$$\psi(z_1)\psi(z_2) \sim \frac{1}{z_1 - z_2} \quad (11)$$

i.e., the OPE of ψ does not generate any new operators. Thus the center algebra for the Pfaffian wave function is given by

$$1 \times 1 \sim 1, \quad 1 \times \psi \sim \psi, \quad \psi \times \psi \sim 1. \quad (12)$$

We would like to point out that the structure of the operator algebra generated by ψ is very important. This structure reflects the structure of internal correlation of the wave function. Instead of directly working with the wave function, we can study the internal structure of FQH state by studying the associated center algebra. The topological order of a quantum Hall wave function can be determined by the center algebra. This is the heart of the operator (or CFT) approach to the FQH wave function.

The CFT description is very useful in the study of the properties of quasiparticles. To construct the quasiparticles in the p-wave paired FQH state, we need to extend the center algebra generated by ψ to include the disorder operator σ , which is known to have dimension 1/16 in the Ising model. The extended operator algebra is generated by ψ and σ which has the following fusion rules

$$\begin{aligned} \psi \times \psi &\sim 1, & \sigma \times \sigma &\sim 1, & \nu \times \nu &\sim 1, \\ \psi \times \sigma &\sim \nu, & \psi \times \nu &\sim \sigma, & \sigma \times \nu &\sim \psi. \end{aligned} \quad (13)$$

The quasiparticle excitations can be expressed in terms of the known Ising correlations of the σ operators, which in turn enable us to determine quantum numbers of the quasiparticles. Detailed discussion can be found in Ref. 7. Note that any new disorder operator in the extended algebra must be semilocal with respect to ψ , so that one can use them (together with appropriate U(1) part) to construct single-valued electron wave functions.^{7,5}

3. THE D-WAVE PAIRED STATE FOR SPINLESS ELECTRONS

The next-to-simplest example is the d-wave paired FQH state for spinless electrons, which is a natural generalization of the Pfaffian or p-wave paired state. The total wave function for this state is of a form similar to that of (7):

$$\begin{aligned} \Phi &= \Phi_d \Phi_m \\ \Phi_d &= \mathcal{S}\left(\frac{1}{(z_1 - z_2)^2} \frac{1}{(z_3 - z_4)^2} \dots\right) \\ \Phi_m &= \left(\prod_{i < j} (z_i - z_j)^m\right) e^{-\frac{1}{4} \sum_i |z_i|^2} \end{aligned} \quad (14)$$

The differences with (7) are: first, the negative power of paired $z_i - z_j$ in Φ_d is two rather than one and, secondly, the symmetrization \mathcal{S} replaces the antisymmetrization \mathcal{A} . Thus m must be an odd integer, to make the total wave function Φ anti-symmetric.

Let us first analyze the structure of zeros of this wave function assuming, for simplicity, $m = 3$. Let $z_1 = z_3 + \delta_1$ and $z_2 = z_3 + \delta_2$, we find Φ has the following expansion

$$\Phi = \sum_{k=\text{odd}} (\delta_2)^k \sum_l (\delta_1)^l A_{kl}(z_3, z_4, \dots) \quad (15)$$

One can directly check that the coefficients

$$A_{12} = A_{14} = A_{33} = 0 \quad (16)$$

Therefore Φ is the exact ground state of Hamiltonian H with the following three-body potential

$$\begin{aligned} V = & -V_1 \partial_{z_2^*} \delta(z_2 - z_3) \partial_{z_2} \partial_{z_1^*}^2 \delta(z_1 - z_3) \partial_{z_1}^2 \\ & - V_2 \partial_{z_2^*} \delta(z_2 - z_3) \partial_{z_2} \partial_{z_1^*}^4 \delta(z_1 - z_3) \partial_{z_1}^4 + V_3 \partial_{z_2^*}^3 \delta(z_2 - z_3) \partial_{z_2}^3 \partial_{z_1^*}^3 \delta(z_1 - z_3) \partial_{z_1}^3 \end{aligned} \quad (17)$$

This is because the Hamiltonian H is positive definite if $V_i > 0$, and Φ is a zero-energy state of H . We have checked numerically that Φ is indeed the non-degenerate ground state of H (on a sphere) with a finite energy gap. This implies that the constraint (16) uniquely fixes the wave function.

Now the two-point function is $\Phi_d = 1/(z_1 - z_2)^2$. Thus ψ is of dimension $h = 1$. Also Φ_d has the property that as $z_1 \rightarrow z_2$,

$$\Phi_d(z_1, \dots) = \frac{\Phi_d(z_3, \dots)}{(z_1 - z_2)^2} + O(1)$$

Again there are no terms of one power of $z_1 - z_2$ higher than the leading singular term. So (6) makes perfect sense and can be used to define the energy-momentum tensor. This leads to the identification

$$\langle T(z_2) \prod_{i=3} \psi(z_i) \rangle \equiv \frac{c}{2} \lim_{z_1 \rightarrow z_2} \left(\Phi_d(z_1, z_2, \dots) - \frac{1}{(z_1 - z_2)^2} \Phi_d(z_3, z_4, \dots) \right). \quad (18)$$

Thus the conformal Ward identities (2) and (4) are reduced to the following conditions on the wave function Φ_d :

$$\begin{aligned} & c/2 \lim_{z_1 \rightarrow z_2} \left(\Phi_d(z_1, z_2, \dots) - \frac{1}{(z_1 - z_2)^2} \Phi_d(z_3, z_4, \dots) \right) \\ & = \sum_{k=3} \left(\frac{1}{(z_2 - z_k)^2} + \frac{1}{z_2 - z_k} \partial_{z_k} \right) \Phi_d(z_3, z_4, \dots) \end{aligned} \quad (19)$$

and

$$\begin{aligned}
& \frac{c^2}{4} \lim_{z_1 \rightarrow z_2, z_3 \rightarrow z_4} \left(\Phi_d(z_1, z_2, \dots) - \frac{\Phi_d(z_3, z_4, \dots)}{(z_1 - z_2)^2} - \frac{\Phi_d(z_1, z_2, z_5 \dots)}{(z_3 - z_4)^2} \right. \\
& \quad \left. + \frac{\Phi_d(z_5, z_6 \dots)}{(z_1 - z_2)^2 (z_3 - z_4)^2} \right) \\
&= \sum_{k=5} \left(\frac{1}{(z_2 - z_k)^2} + \frac{1}{z_2 - z_k} \partial_{z_k} \right) \langle T(z_4) \prod_{i=5} \psi(z_i) \rangle \\
& \quad + \left(\frac{2}{(z_2 - z_4)^2} + \frac{1}{z_2 - z_4} \partial_{z_4} \right) \langle T(z_4) \prod_{i=5} \psi(z_i) \rangle + \frac{c/2}{(z_2 - z_4)^4} \Phi_d(z_5, z_6 \dots)
\end{aligned} \tag{20}$$

where $\langle T(z_4) \prod_{i=5} \psi(z_i) \rangle$ is given by (18). One can directly check that both (19) and (20) are indeed satisfied by the wave function Φ_d if we choose the central charge $c = 1$.

This implies that Φ_d can be viewed as a correlation of the dimension-one ($h = 1$) primary field ψ in a $c = 1$ CFT. Note that these values of the pair (h, c) satisfies the Kac formula with $n = 3, m = 1$:

$$h_{(n,m)} = h_0 + \frac{1}{4}(\alpha_+ n + \alpha_- m)^2, \tag{21}$$

where

$$h_0 = \frac{1}{24}(c - 1), \quad \alpha_{\pm} = \frac{\sqrt{1 - c} \pm \sqrt{25 - c}}{\sqrt{24}}. \tag{22}$$

Thus ψ is a degenerate primary field, whose correlation (or the wave function Φ_d) should satisfy a third-order differential equation¹⁵:

$$\begin{aligned}
& \left\{ \frac{1}{2} \frac{\partial^3}{\partial z^3} - \sum_{i=1} \frac{2}{(z - z_i)^3} - \sum_{i=1} \frac{1}{(z - z_i)^2} \frac{\partial}{\partial z_i} \right. \\
& \quad \left. - \sum_{i=1} \frac{2}{(z - z_i)^2} \frac{\partial}{\partial z} - \sum_{i=1} \frac{2}{(z - z_i)} \frac{\partial^2}{\partial z \partial z_i} \right\} \Phi_d(z, z_1, z_2, \dots) = 0.
\end{aligned} \tag{23}$$

Indeed, one can explicitly verify that this equation is satisfied by Φ_d given by (18). This further confirms that Φ_d can be written as correlation in a $c = 1$ CFT. In fact we have directly checked that Φ_d is the correlation of the $U(1)$ current in the Gaussian model.

Introducing the electron operator

$$\psi_e(z) = \psi(z) e^{i\sqrt{m}\phi(z)} \tag{24}$$

we find that the wave function Φ in (14) can be written as

$$\Phi = \left\langle \prod_i \psi_e(z_i) e^{-i\frac{1}{\sqrt{m}} \int d^2z \phi} \right\rangle \tag{25}$$

Now the constraint (16) on zeros of Φ becomes a consequence of the OPE of ψ_e . Let $z_1 = z_3 + \delta_1$ and $z_2 = z_3 + \delta_2$ and assume $m = 3$, we find

$$\begin{aligned} & \psi_e(z_1)\psi_e(z_2)\psi_e(z_3) \\ &= \psi_e(z_1) \left(\delta_2 e^{2i\sqrt{m}\phi(z_3)} + (\delta_2)^3 2T(z_3) e^{2i\sqrt{m}\phi(z_3)} \right) + \dots \\ &= \delta_2 (\delta_1)^6 \psi(z_3) e^{3i\sqrt{m}\phi(z_3)} + (\delta_2)^3 (\delta_1)^4 2\psi(z_3) e^{3i\sqrt{m}\phi(z_3)} + \dots \end{aligned} \quad (26)$$

The lower-order zeros are absent as described by (16).

Now let us discuss quasiparticle excitations in the d-wave paired state. First we notice that the following operator algebra

$$\begin{aligned} \partial_{\bar{z}}\psi &= 0 \\ [\psi(z_1), \psi(z_2)] &= 0 \\ \psi(z)\psi(0) &= \frac{1}{z^2} + O(1) \end{aligned} \quad (27)$$

completely determines the correlations between ψ . This is because the algebra (27) completely determines the poles and their residues in the correlation. Being a holomorphic function, the correlation is thus uniquely determined and turns out to be none other than Φ_d .

To study the quasiparticles in this FQH state, we need to extend the operator algebra (27) to include the disorder operator η . We require the correlation between ψ and η to acquire a minus sign as we move ψ around η . Other phases are not allowed, because $\psi^2 \sim 1$ and moving a pair of ψ around η should not give any phase. Thus we may try the following OPE between ψ and η :

$$\psi(z)\eta(0) \sim z^{-\frac{1}{2}}(\tilde{\eta}(0) + O(z)) \quad (28)$$

where $\tilde{\eta}$ is some other operator.

To calculate the correlations between ψ 's and two η operators¹⁸ we note that, as a consequence of (27) and (28),

$$\prod_{m=1,2} (z_1 - u_m)^{1/2} \langle \prod_{i=1} \psi(z_i) \prod_{m=1,2} \eta(u_m) \rangle \quad (29)$$

is a holomorphic function of z_1 which has only poles at $z_1 = z_i$, $i = 2, 3, \dots$. We also note that, as a function of z_1 ,

$$\Phi_q(z_1..; u_1, u_2) \equiv \langle \prod_{i=1} \psi(z_i) \prod_{m=1,2} \eta(u_m) \rangle \quad (30)$$

has a pure second-order pole (no first-order pole) at $z_1 = z_i$ with a residue $\Phi_q(\hat{z}_1, \hat{z}_i)$, where $\Phi_q(\hat{z}_1, \hat{z}_i)$ is $\Phi_q(z_1..; u_1, u_2)$ with variables z_1 and z_i removed. This implies that the second-order pole at $z_1 = z_i$ in (29) has a residue $\Phi_q(\hat{z}_1, \hat{z}_i) \prod_{m=1,2} (z_i - u_m)^{1/2}$ and the first-order pole has a residue $\partial_{z_1} \Phi_q(\hat{z}_1, \hat{z}_i) \prod_{m=1,2} (z_1 - u_m)^{1/2}$ evaluated at $z_1 = z_i$. We

obtain the following relation for Φ_q :

$$\begin{aligned} & \Phi_q(z_{1..}; u_1, u_2) \\ &= \prod_{m=1,2} (z_1 - u_m)^{-1/2} \sum_{k=2} \left(\frac{1}{(z_1 - z_k)^2} + \frac{1}{2(z_1 - z_k)} \sum_{m=1,2} \frac{1}{z_k - u_m} \right) \\ & \times \prod_{m=1,2} (z_k - u_m)^{1/2} \Phi_q(\hat{z}_1, \hat{z}_k) \end{aligned} \quad (31)$$

(31) relates a correlation with $2N$ ψ 's to one with $2(N-1)$ ψ 's. Thus from (31) we can calculate any correlations between ψ 's and η 's from those between only η 's.

We would like to point out that (31) is actually over-determined. It is crucial to check the self consistency of (31), namely Φ_q calculated from different decomposition paths (or different ways of reduction) agree with each other. To show (31) is self consistent, we may rewrite it as

$$\Phi_q(z_{1..}; u_1, u_2) = \sum_{k=2} f_{u_1, u_2}(z_1, z_k) \Phi_q(\hat{z}_1; \hat{z}_k) \quad (32)$$

where

$$f_{u_1, u_2}(z, w) = \frac{1}{(z-w)^2} \frac{zw + u_1 u_2 - \frac{1}{2}(z+w)(u_1 + u_2)}{\prod_{m=1,2} \left((z - u_m)^{-1/2} (w - u_m)^{-1/2} \right)} \quad (33)$$

(32) has a unique solution

$$\Phi_q(z_{1..}; u_1, u_2) = \mathcal{S}(f_{u_1, u_2}(z_1, z_2) f_{u_1, u_2}(z_3, z_4) \dots) \frac{1}{(u_1 - u_2)^{2h_\eta}} \quad (34)$$

where h_η is the dimension of η . We have explicitly verified that the four-point correlation $\langle \psi(z_1) \psi(z_2) \eta(u_1) \eta(u_2) \rangle$ with $z_1 \rightarrow z_2$ satisfies the conformal Ward identity if we choose $h_\eta = 1/16$. This implies that η is a dimension-1/16 primary field. In fact η is just the sector-changing operator in a $c = 1$ Z_2 -orbifold model^{19,20}. Such an operator is a dimension-1/16 primary field, whose OPE with the $U(1)$ current (which is identified as ψ) has the same structure as that in (28). With this identification it is not hard to convince oneself that Φ_q satisfies the conformal Ward identity.

Using η we can construct the quasihole excitations in the d-wave paired state:

$$\Phi_m(\{z_i\}) \Phi_q(\{z_i\}; u_1, u_2) \prod_k \prod_{m=1,2} (z_k - u_m)^{1/2} \quad (35)$$

From (34) we can see that (35) is a single-valued and finite function of the electron coordinates z_i . The wave function describes quasiholes located at u_m . The quasihole carries a fractional charge $1/2m$ in units of electronic charge. This is because the bound state of two such quasiholes becomes the charge- $1/m$ quasihole created by inserting a unit flux:

$$\Phi_m(z_i) \Phi_d(z_i) \prod_k (z_k - u) \quad (36)$$

since $\lim_{u_1 \rightarrow u_2} \Phi_q(z_i; u_1, u_2) \propto \Phi_d(z_i)$, as one can see from (34) and (33).

Notice that (35) can be written as a correlation between primary fields

$$\langle \prod_{m=1,2} \psi_q(u_m) \prod_i \psi_e(z_i) e^{-i \int d^2 z \phi(z)} \rangle \quad (37)$$

where the quasihole operator

$$\psi_q = \eta e^{i \frac{1}{2\sqrt{m}} \phi} \quad (38)$$

Thus as a function of z_i , (37) has the same local structure of zeros as the ground state wave function (14). The structure of zeros is determined by the OPE (26) and satisfies (16) if $m = 3$. Thus (37) is a zero-energy eigenstate of H in (17). We would like to stress that, when we use primary fields to create quasiparticle excitations, the structure of zeros in the electron wave function is not changed. This suggests that the quasiparticle created by a primary field is a local excitation. The above analysis also implies that the conformal blocks generated by the quasiparticle operators correspond to degenerate states (with zero energy). Generally the degenerate states induce non-abelian Berry's phases as we interchange quasiparticles, which correspond to the non-abelian statistics of the quasiparticles.

4. THE HALDANE-REZAYI STATE

One candidate for the $\nu = 5/2$ FQH state observed in experiments²¹ is the Haldane-Rezayi (HR) state.²² The HR state is a d-wave-paired spin-singlet FQH state for spin-1/2 electrons:

$$\begin{aligned} \Phi_{HR}(z_i, w_i) &= \Phi_m(z_i, w_i) \Phi_{ds}(z_i, w_i) \\ \Phi_{ds}(z_i, w_i) &= \mathcal{A}_{z,w} \left(\frac{1}{(z_1 - w_1)^2} \frac{1}{(z_2 - w_2)^2} \dots \right) \\ \Phi_m(z_i, w_i) &= \left(\prod_{i < j} (z_i - z_j)^m \prod_{i < j} (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^m \right) e^{-\frac{1}{4} \sum_i (|z_i|^2 + |w_i|^2)} \end{aligned} \quad (39)$$

which has a filling fraction $1/m$ with m an even integer. ($m = 2$ at the second Landau level gives rise to $\nu = 2 + 1/2 = 5/2$.) Here z_i (w_i) are the coordinates of the spin-up (-down) electrons, and the operator $\mathcal{A}_{z,w}$ performs separate antisymmetrizations among z_i 's and among w_i 's. One can directly check that Φ_{ds} is indeed a spin singlet, and Φ_m , when viewed as an operator, commutes with the total spin operator.

Let us first analyze the structure of zeros in the above wave function assuming, for simplicity, $m = 2$. Let $z_1 = z_2 + \delta_1$ and $w_1 = w_2 + \delta_2$, we find that in the expansion of Φ_{HR} ,

$$\begin{aligned} \Phi_{HR} &= \sum_{l=odd} (\delta_1)^l A_l(z_2, \dots; w_1, \dots) \\ \Phi_{HR} &= \sum_n (\delta_2)^n B_n(z_2, \dots; w_1, \dots) \end{aligned} \quad (40)$$

the term linear in δ_1 or δ_2 is absent:

$$A_1 = B_1 = 0. \quad (41)$$

Therefore Φ is the exact ground state of the following two-body Hamiltonian

$$H = -V_1 \partial_{z_1^*} \delta(z_1 - z_2) \partial_{z_1} - V_2 \partial_{z_1^*} \delta(z_1 - w_1) \partial_{z_1} \quad (42)$$

since H is positive definite for $V_i > 0$, and Φ_{HR} has a zero average energy. It has been checked numerically that Φ_{HR} is the unique incompressible ground state of H .²²

Again we would like to know whether we can represent the pairing wave function Ψ_{ds} as a correlation of two operators ψ_{\pm} :²³

$$\Phi_{ds}(z_i, w_i) = \langle \prod_i (\psi_+(z_i) \psi_-(w_i)) \rangle \quad (43)$$

Since $\Phi_{ds}(z_1, w_1) \neq 0$, this motivates us to ask a simpler question: using a CFT that contains

$$\psi_+(z) \psi_-(w) = \frac{1}{(z_1 - z_2)^{2h}} \left(1 + \frac{2h}{c} (z_1 - z_2)^2 T(z_2) + O((z_1 - z_2)^3) \right) \quad (44)$$

can we reproduce the wave function Φ_{ds} ? To answer this question, we need to introduce the energy-momentum tensor

$$T(z) = \frac{c}{2h} \lim_{z_1 \rightarrow z} \left(\psi_+(z_1) \psi_-(z) - \frac{1}{(z_1 - z)^{2h}} \right) (z_1 - z)^{2h-2} \quad (45)$$

and to check that the wave function satisfies the Ward identity (2) and T satisfies the Virasoro algebra (4). Note as $z_1 \rightarrow w_1$, Φ_{ds} has an expansion

$$\Phi_{ds}(z_1 \dots; w_1 \dots) = \frac{\Phi_{ds}(z_2 \dots; w_2 \dots)}{(z_1 - w_1)^2} + O(1)$$

The absence of the $1/(z_1 - w_1)$ term implies that (45) gives us a well-defined operator.

From the two point function $\Phi_{ds}(z, w) = 1/(z - w)^2$, we find the dimension of the ψ_{\pm} is $h = 1$. By taking proper limits of the pairing wave function, we show in appendix that Φ_{ds} indeed satisfies the Ward identity (2) and the Virasoro algebra (4), if the central charge c is taken to be $c = -2$.

This result implies that Φ_{ds} can be written as a correlation of two dimension-one operators in a $c = -2$ CFT. The $c = -2$ CFT is closely related to a (non-unitary) minimal model, since the central charge satisfies the condition for a minimal model¹⁵

$$\frac{\sqrt{25 - c} - \sqrt{1 - c}}{\sqrt{25 - c} + \sqrt{1 - c}} = \frac{p}{q} \quad (46)$$

with $p = 1$ and $q = 2$. Furthermore, ψ_{\pm} (with dimension $h = 1$) are degenerate primary fields of level two¹⁵, since the pair $(h, c) = (1, -2)$ satisfies the Kac formula (21) with

$n = 2$ and $m = 1$. As a consequence, Φ_{ds} (as a correlation of ψ_{\pm} in CFT) should satisfy a second-order partial differential equation:

$$\left[\frac{1}{2} \partial_{z_1}^2 - \sum_{i=2} \left(\frac{1}{(z_1 - z_i)^2} + \frac{1}{z_1 - z_i} \partial_{z_i} \right) - \sum_{i=1} \left(\frac{1}{(z_1 - w_i)^2} + \frac{1}{z_1 - w_i} \partial_{w_i} \right) \right] \Phi_{ds}(z_i, w_i) = 0 \quad (47)$$

and similar equations for z_2, z_3, \dots and w_1, w_2, \dots . We have explicitly checked that Φ_{ds} indeed satisfies the differential equation (47). This further confirms that Φ_{ds} can be viewed as correlation of degenerate primary fields in a $c = -2$ CFT.

To understand the operator algebra generated from the OPE of ψ_{\pm} , we first notice that the energy-momentum tensor T is invariant under the $SU(2)$ rotation of the electron spin. This is because $f(\xi; z_i, w_i) \equiv \langle T(\xi) \prod_i \psi_+(z_i) \psi_-(w_i) \rangle$ is a spin singlet when viewed as an electron wave function (see Appendix). Therefore our $c = -2$ CFT has an $SU(2)$ symmetry, and the primary fields can be arranged into $SU(2)$ multiplets, ψ_m^s , where s is the total spin and m the S_z quantum number. In this notation $\psi_{\pm} = \psi_{\pm 1/2}^{1/2}$. Let us consider the leading term in the OPE of two ψ_+ operators. The $SU(2)$ symmetry allows us to write

$$\psi_+(z) \psi_+(0) = C_{\frac{1}{2}\frac{1}{2}}^1 z^{h_1 - 2h_{1/2}} \psi_1^1(0) + \dots \quad (48)$$

where $h_{1/2} = 1$ is the dimension of ψ_{\pm} , h_1 and $C_{\frac{1}{2}\frac{1}{2}}^1$ are two constants. In CFT, OPE of any two primary fields must be nonzero and the leading term must be a primary field. Therefore ψ_1^1 is a spin-1 primary field with dimension h_1 . The $SU(2)$ symmetry implies that there exist two other primary fields ψ_0^1 and ψ_{-1}^1 of the same dimension. By taking the limit $z_1 \rightarrow z_2$ in the wave function Φ_{sd} , we find the leading term is proportional to $(z_1 - z_2)$. Thus $h_1 - 2h_{1/2} = 1$ and $h_1 = 3$. Knowing the correlation of ψ_{\pm} and the OPE (48), we can calculate any correlations between ψ_1^1 and ψ_{\pm} . This, in turn, allows us to calculate the OPE between ψ_1^1 and ψ_+ :

$$\psi_+(z) \psi_1^1(0) = C_{\frac{1}{2}1}^{\frac{3}{2}} z^{h_{3/2} - h_{1/2} - h_1} \psi_{3/2}^{3/2}(0) + \dots \quad (49)$$

Since Φ_{ds} is antisymmetric in z_i , we have the expansion $\Phi_{ds}(z_i, w_i) \propto (z_1 - z_2)(z_2 - z_3)(z_3 - z_1) + \dots$ as $z_1, z_2 \rightarrow z_3$. This implies that $\langle \psi_+(z) \psi_1^1(0) \dots \rangle \sim z^2$ as $z \rightarrow 0$ and $h_{3/2} = 6$. In general the OPE of ψ_{\pm} generate operators with all possible values of spin. The chiral operator algebra (the center algebra) generated by ψ_{\pm} contains (at least) primary fields ψ_m^s , $s = 1/2, 1, 3/2, \dots$ and $m = -s, -s + 1, \dots, s$. The dimension h_s of the spin- s operator ψ_m^s satisfies $h_{s+\frac{1}{2}} - h_s - h_{\frac{1}{2}} = 2s$, which implies

$$h_s = \frac{(4s - 1)^2 - 1}{8} \quad (50)$$

(50) tells us that ψ_m^s can be identified as the level- $(2s + 1)$ degenerate operator $\psi_{(2s+1,1)}$ in the $c = -2$ model. (Here we use the notation in Ref. 15.) In contrast to other minimal models, our $c = -2$ CFT contains infinite number of primary fields and has a global $SU(2)$ symmetry.

The operator algebra of ψ_m^s fields determines the correlations between those operators. However to calculate the correlations between ψ_\pm we do not need to know the full structure of the operator algebra. In fact the following operator algebra

$$\begin{aligned} \partial_{\bar{z}}\psi_\pm &= 0 \\ \{\psi_a(z_1), \psi_a(z_2)\} &= 0, \quad a = +, - \\ \psi_+(z)\psi_-(0) &= \frac{1}{z^2} + O(1) \end{aligned} \tag{51}$$

completely determines the correlations between ψ_\pm , since the algebra (51) completely determines the poles and their residues in the correlation.

Now let us introduce the electron operator

$$\psi_{e\pm}(z) = \psi_\pm(z)e^{i\sqrt{m}\phi(z)}. \tag{52}$$

Then the wave function Φ_{HR} in (39) can be written as

$$\Phi_{HR} = \left\langle \prod_i [\psi_{e+}(z_i)\psi_{e-}(w_i)] e^{-i\frac{1}{\sqrt{m}} \int d^2z \phi} \right\rangle, \tag{53}$$

and the constraint (41) on zeros of Φ_{HR} can be derived from the OPE of $\psi_{e\pm}$. Let $z_1 = z_2 + \delta_1$ and $z_1 = w_1 + \delta_2$ and assume $m = 2$; we find

$$\begin{aligned} \psi_{e+}(z_1)\psi_{e+}(z_2) &= (\delta_1)^3 \psi_1^1(z_2) e^{2i\sqrt{m}\phi(z_2)} \\ \psi_{e+}(z_1)\psi_{e-}(w_1) &= e^{2i\sqrt{m}\phi(w_1)} + O((\delta_2)^2) \end{aligned} \tag{54}$$

The absence of terms linear in δ_1 and δ_2 implies (41).

To study the quasiparticles in the HR state, we need to extend the operator algebra (51) to include the disorder operator η . The $c = -2$ CFT contains another level-2 degenerate primary field $\psi_{(1,2)}$ which has a dimension $-1/8$. It is natural to identify this operator as the disorder operator η . From the fusion rule in the $c = -2$ model $\psi_{(2,1)}(z)\psi_{(1,2)}(0) \sim z^{-1/2}\psi_{(2,2)}(0)$, we find

$$\psi_\pm(z)\eta(0) \sim z^{-1/2}(\tilde{\eta}_\pm(0) + O(z)) \tag{55}$$

Note that as we move ψ_\pm around η , the correlation obtains a minus sign which commutes with the $SU(2)$ spin rotation. Thus we expect η to be a spin-singlet operator.

To calculate the correlations between ψ_\pm and two η 's¹⁸ we note that again

$$\prod_{m=1,2} (z_1 - u_m)^{1/2} \left\langle \prod_{i=1} (\psi_+(z_i)\psi_-(w_i)) \prod_{m=1,2} \eta(u_m) \right\rangle \tag{56}$$

is a holomorphic function of z_1 which has only poles at $z_1 = w_i$. We also note that, as a function of z_1 ,

$$\Phi_q(z_1 \dots; w_1 \dots; u_1, u_2) \equiv \left\langle \prod_{i=1} (\psi_+(z_i)\psi_-(w_i)) \prod_{m=1,2} \eta(u_m) \right\rangle \tag{57}$$

has a pure second-order pole (no first order pole) at $z_1 = w_k$ with a residue $(-)^{k+1}\Phi_q(\hat{z}_1; \hat{w}_k)$, where $\Phi_q(\hat{z}_1; \hat{w}_k)$ is $\Phi_q(z_{1..}; w_{1..}; u_1, u_2)$ with variables z_1 and w_k removed. This implies that the second-order pole at $z_1 = w_k$ in (56) has a residue $(-)^{k+1}\Phi_q(\hat{z}_1; \hat{w}_k) \prod_{m=1,2}(w_k - u_m)^{1/2}$ and the first-order pole has a residue $\partial_{z_1}(-)^{k+1}\Phi_q(\hat{z}_1; \hat{w}_k) \prod_{m=1,2}(z_1 - u_m)^{1/2}$ evaluated at $z_1 = w_k$. Thus we obtain the following relation between correlations Φ_q :

$$\begin{aligned} & \Phi_q(z_{1..}; w_{1..}; u_1, u_2) \\ &= \prod_{m=1,2} (z_1 - u_m)^{-1/2} \sum_k (-)^{k+1} \left(\frac{1}{(z_1 - w_k)^2} + \frac{1}{2(z_1 - w_k)} \sum_{m=1,2} \frac{1}{w_k - u_m} \right) \\ & \times \prod_{m=1,2} (w_k - u_m)^{1/2} \Phi_q(\hat{z}_1; \hat{w}_k) \end{aligned} \quad (58)$$

It relates a correlation with N pairs of ψ_{\pm} to the one with $N - 1$ pairs, and determines any correlations between ψ_{\pm} 's and two η 's from those between only η 's.

Again (58) is over determined and it is crucial to check the self-consistency of (58). In fact (58) can be rewritten as

$$\Phi_q(z_{1..}; w_{1..}; u_1, u_2) = \sum_k (-)^{k+1} f_{u_1, u_2}(z_1, w_k) \Phi_q(\hat{z}_1; \hat{w}_k) \quad (59)$$

where $f_{u_1, u_2}(z, w)$ is given in (33). (59) has a unique solution

$$\Phi_q(z_{1..}; w_{1..}; u_1, u_2) = \mathcal{A}_{z, w} (f_{u_1, u_2}(z_1, w_1) f_{u_1, u_2}(z_2, w_2) \dots) \frac{1}{(u_1 - u_2)^{2h_{\eta}}} \quad (60)$$

Since $f_{u_1, u_2}(z, w) = f_{u_1, u_2}(w, z)$, we see that Φ_q is a spin singlet, which implies that η is a spin-singlet operator, as we have expected. In the Appendix we will show that Φ_q satisfies the conformal Ward identity if we choose $h_{\eta} = -1/8$. Thus η is a dimension- $(-1/8)$ primary field, which agrees with general considerations for the $c = -2$ CFT.

Using η , we can write down the wave function of the quasihole excitations in the HR state:

$$\Phi_m(\{z_i; w_j\}) \Phi_q(\{z_i; w_j\}; u_1, u_2) \prod_k \prod_{m=1,2} (z_k - u_m)^{1/2} (w_k - u_m)^{1/2} \quad (61)$$

From (60) we can see that (61) is single-valued and non-singular as a function of the electron coordinates z_i and w_j . The wave function describes quasiholes whose locations are described by the parameters u_m . The quasihole carries $1/2m$ unit of electron charge, which is induced by the factor $\prod_{k,m} ((z_k - u_m)^{1/2} (w_k - u_m)^{1/2})$. The bound state of two such quasiholes becomes the charge- $1/m$ quasihole:

$$\Phi_m(z_i; w_j) \Phi_{ds}(z_i; w_j) \prod_k (z_k - u)(w_k - u) \quad (62)$$

since $\lim_{u_1 \rightarrow u_2} \Phi_q(z_i; w_j; u_1, u_2) \propto \Phi_{ds}(z_i; w_j)$ (see (60)). Again since the quasiholes are created by insertion of primary fields, the quasihole wave function (61) is a zero-energy eigenstate of H in (42). The system also contains neutral spin- $1/2$ excitations which can be viewed as a bound state of an electron and m charge- $1/m$ quasiholes. Other excitations can be viewed as bound states of charge $1/2m$ spin-singlet excitations and neutral spin- $1/2$ excitations.

5. D-WAVE-PAIRED SPIN-TRIPLET FQH STATE

The d-wave-paired spin-triplet FQH state for spin-1/2 electrons is given by:

$$\begin{aligned}\Phi_{DL}(z_i, w_i) &= \Phi_{lmn}(z_i, w_i) \Phi_{dt}(z_i, w_i) \\ \Phi_{dt}(z_i, w_i) &= \mathcal{S}_{z,w} \left(\frac{1}{(z_1 - w_1)^2} \frac{1}{(z_2 - w_2)^2} \cdots \right) \\ \Phi_{lmn}(z_i, w_i) &= \left(\prod_{i < j} (z_i - z_j)^l \prod_{i < j} (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^n \right) e^{-\frac{1}{4} \sum_i (|z_i|^2 + |w_i|^2)}\end{aligned}\tag{63}$$

which has a filling fraction $\frac{l+m-2n}{lm-n^2}$ with l, m odd integers. Here z_i (w_i) are the coordinates of the spin-up (-down) electrons, and the operator $\mathcal{S}_{z,w}$ performs separate symmetrizations between z_i 's and between w_i 's. Φ_{DL} is not a spin singlet and there is no spin rotation symmetry. It is more natural to view Φ_{DL} as a wave function for double layer FQH systems which involve interlayer pairing. Now z_i and w_i are electron coordinates in the two layers. We will call the FQH state in (63) the DL state.

Again we would like to know whether we can represent the pairing wave function Ψ_{dt} as a correlation of two primary fields ψ_{\pm} :

$$\Phi_{dt}(z_i, w_i) = \langle \prod_i (\psi_+(z_i) \psi_-(w_i)) \rangle\tag{64}$$

Note Φ_{dt} has the following expansion as $z_1 \rightarrow w_1$

$$\Phi_{dt}(z_1 \dots; w_1 \dots) = \frac{\Phi_{dt}(z_2 \dots; w_2 \dots)}{(z_1 - w_1)^2} + O(1)$$

This motivates us to assume

$$\psi_+(z) \psi_-(w) = \frac{1}{(z_1 - z_2)^{2h}} \left(1 + \frac{2h}{c} (z_1 - z_2)^2 T(z_2) + O((z_1 - z_2)^3) \right)\tag{65}$$

with the dimension of ψ_{\pm} being $h = 1$. Thus we may introduce the energy-momentum tensor through

$$T(z) = \frac{c}{2h} \lim_{z_1 \rightarrow z} \left(\psi_+(z_1) \psi_-(z) - \frac{1}{(z_1 - z)^{2h}} \right) (z_1 - z)^{2h-2}\tag{66}$$

We have directly checked that the wave function Φ_{dt} satisfies the Ward identity (2) and T satisfies the Virasoro algebra (4) if we choose $c = 2$. Therefore the Φ_{dt} can be written as a correlation of a dimension-1 operator in a $c = 2$ conformal theory. In fact, one may consider a $U(1) \times U(1)$ Gaussian model which has $c = 2$. Let

$$j_{\pm} = \frac{1}{\sqrt{2}} (j_1 \pm i j_2)\tag{67}$$

where j_1 and j_2 are the currents of the two $U(1)$ models. One can easily check that the correlations of j_{\pm} reproduce Φ_{dt} . Thus we can identify ψ_{\pm} as j_{\pm} .

The wave function Φ_{dt} can be derived from the following operator algebra

$$\begin{aligned} \partial_{\bar{z}}\psi_{\pm} &= 0 \\ [\psi_a(z_1), \psi_a(z_2)] &= 0, \quad a = +, - \\ \psi_+(z)\psi_-(0) &= \frac{1}{z^2} + O(1) \end{aligned} \quad (68)$$

To study the quasiparticles in the DL state, we need to extend the operator algebra (68) to include the disorder operator η . Again let us try

$$\psi_{\pm}(z)\eta(0) \sim z^{-1/2}(\tilde{\eta}_{\pm}(0) + O(z)) \quad (69)$$

The correlations Φ_q between ψ_{\pm} and two η 's¹⁸ can be calculated from (68) and (69). We find that Φ_q satisfies

$$\begin{aligned} &\Phi_q(z_1..; w_1..; u_1, u_2) \\ &= \prod_{m=1,2} (z_1 - u_m)^{-1/2} \sum_k \left(\frac{1}{(z_1 - w_k)^2} + \frac{1}{2(z_1 - w_k)} \sum_{m=1,2} \frac{1}{w_k - u_m} \right) \\ &\quad \times \prod_{m=1,2} (w_k - u_m)^{1/2} \Phi_q(\hat{z}_1; \hat{w}_k) \end{aligned} \quad (70)$$

Again (70) can be rewritten as

$$\begin{aligned} &\Phi_q(z_1..; w_1..; u_1, u_2) \\ &= \sum_k f_{u_1, u_2}(z_l, w_k) \Phi_q(\hat{z}_l; \hat{w}_k) \\ &= \mathcal{S}_{z, w} (f_{u_1, u_2}(z_1, w_1) f_{u_1, u_2}(z_2, w_2) \dots) \frac{1}{(u_1 - u_2)^{2h_{\eta}}} \end{aligned} \quad (71)$$

where $f_{u_1, u_2}(z, w)$ is given in (33). The fact that (71) has a unique solution implies that the OPE (68) and (69) are self consistent. One can show that Φ_q satisfies the conformal Ward identity if we choose $h_{\eta} = 1/8$. This implies that η is a dimension-1/8 primary field. Again η can be identified as the sector-changing operator in the $c = 2$ orbifold model.

The quasihole excitations in the DL state is given by

$$\Phi_m(\{z_i; w_j\}) \Phi_q(\{z_i; w_j\}; u_1, u_2) \prod_k \prod_{m=1,2} (z_k - u_m)^{1/2} (w_k - u_m)^{1/2} \quad (72)$$

(72) is a single-valued and finite function of the electron coordinates z_i and w_j . The wave function describes quasiholes located at u_m which carry electric charge $\nu/2$. The bound state of two such quasiholes becomes the charge- ν quasihole created by inserting a unit flux:

$$\Phi_m(z_i; w_j) \Phi_{dt}(z_i; w_j) \prod_k (z_k - u)(w_k - u) \quad (73)$$

This is because $\lim_{u_1 \rightarrow u_2} \Phi_q(z_i; w_j; u_1, u_2) \propto \Phi_{dt}(z_i; w_j)$ (see (71)).

6. CONCLUSIONS AND DISCUSSIONS

In this paper we have proposed a new approach to explore the relationship between FQHE and CFT. Instead of generating FQH wave functions from correlations in a CFT chosen *a priori*, we start with a certain FQH many-body wave function and ask whether it can be consistently viewed as a correlation in appropriate CFT. Following techniques often used in CFT, we find that by choosing some simple ansatz for the OPE of the electron operator, whose consistency can be checked later, conformal Ward identities can be turned into partial differential equations relating the N -body to less-body wave functions, if the given FQH wave function is to be identified with correlations in a (chiral) CFT. The conformal dimension, h , of the primary field that is to be identified with (the non-Laughlin part of) the electron operator and the central charge, c , of the CFT are determined in the course of verifying the conformal Ward identities. If the pair (h, c) happen to satisfy the Kac formula, the primary field must be a degenerate one and their correlations, according to general principles of CFT, must satisfy certain partial differential equations called the null vector condition. Whether the given FQH wave function satisfies these equations is an independent check for their identification with correlation functions in the CFT. We have explicitly shown that several many-body FQH wave functions, such as p-wave and d-wave paired states for spinless electrons and d-wave paired spin singlet and triplet for spin-1/2 electrons, indeed pass the check for conformal Ward identities and for the null vector condition. We feel that passing through these nontrivial checkings should not be a mere accident; it really implies the existence of a conformal operator product algebra (the center algebra), which is generated by the OPE of the electron operator(s) as primary field, hidden in *these* FQH wave functions. We propose to use this conformal operator algebra to characterize the topological order in the corresponding FQH states. Namely the quantum numbers of the quasiparticle excitations in these FQH states, the ground state degeneracy on a torus as well as the spectrum of edge states should be determined from the knowledge of the associated hidden conformal operator product algebra.

Leaving the discussions on other topological properties to further research, in this paper we have tried only to determine the quantum numbers of quasiparticle excitations. To study quasiparticle excitations we need to consistently extend the operator product algebra that we obtained from the electron operator(s) to include disorder operators. This is a highly nontrivial task in a *chiral* CFT. We have not been able to give a systematic procedure or a criterion for the completeness for such extension. By guesswork and some consistency check, we have been able to include certain simple disorder operators and use them to make predictions on quantum numbers of part of quasiparticles and calculate their wave functions. It seems that a more systematic study needs considerations of how to incorporate more irreducible representations of the operator algebra (the center algebra) generated by the electron operator(s) to form a (minimal) closed fusion-rule algebra. Or one may try to find a bigger underlying operator algebra that organize all states of the FQH system, the ground state and all its excitations, into a single irreducible representation.

Finally we feel it is necessary to make the following cautious remark. It is not clear to us whether every conceivable FQH wave function must be compatible with conformal invariance or conformal algebra. For example, our construction does not apply to f-wave paired state for spinless electrons, though the f-wave paired state is very similar to the p-wave paired state. So we feel the operator algebra generated from the electron operator(s) and its extension by including disorder operators perhaps are more fundamental. Whether the algebra generated by electron (the center algebra) forms a representation of Virasoro algebra or not is not very important. In fact, in above sections we have seen in many cases these operator algebras alone are sufficient to determine relevant wave functions as holomorphic correlations. The existence of the Virasoro algebra simply organizes the operators

in the center algebra into Verma modules and make it easier to study the structure of the center algebra.

We would like to thank Aspen Center of Physics for hospitality, where this work was initiated. XGW is supported by NSF grant No. DMR-91-14553 and YSW by NSF grant No. PHY-9309458. XGW also would like to acknowledge the support from A.P. Sloan Foundation.

Note added: After submission of the present paper we learned that the wave function of the HR state can also be constructed from a scalar fermion theory which has a central charge $c = -2$.²⁴ We would like to thank N. Read for communicating his result prior publication.

APPENDIX

Here we like to show that, for the d-wave spin-singlet wave function Φ_{ds} , the energy-momentum tensor T defined in (45) satisfies the Ward identity (2) and the Virasoro algebra (4) if $c = -2$. The dimension of ψ_{\pm} is taken to be $h = 1$.

Two relations of Φ_{ds} are very useful:

$$\Phi_{ds} = \sum_k (-)^{i+k} \Phi_{ds}(z_i; w_k) \Phi_{ds}(\hat{z}_i; \hat{w}_k) \quad (74)$$

$$\Phi_{ds} = \sum_{k' < k} (-)^{k+k'+1} \Phi_{ds}(z_1, z_2; w_{k'}, w_k) \Phi_{ds}(\hat{z}_1, \hat{z}_2; \hat{w}_{k'}, \hat{w}_k) \quad (75)$$

where $\Phi_{ds}(\hat{z}_i; \hat{w}_k)$ is $\Phi_{ds}(z_1, \dots; w_1, \dots)$ with z_i and w_k variables removed, and Φ_{ds} denotes $\Phi_{ds}(z_1, \dots; w_1, \dots)$.

To check the Ward identity, we first calculate $\langle T(\xi) \prod_i [\psi_+(z_i) \psi_-(w_i)] \rangle$. Consider

$$\Phi_{ds}(\xi + \delta, z_1, \dots; \xi, w_1, \dots) = \delta^{-2} \Phi_{ds} + \sum_k \frac{(-)^k}{(\xi + \delta - w_k)^2} \Phi_{ds}(z_1 \dots; \xi, \hat{w}_k) \quad (76)$$

Taking the limit $\delta \rightarrow 0$ and subtracting out the δ^{-2} term, we find

$$\begin{aligned} & \langle T(\xi) \prod_i [\psi_+(z_i) \psi_-(w_i)] \rangle \\ &= \frac{c}{2} \sum_k \frac{(-)^k}{(\xi - w_k)^2} \Phi_{ds}(z_1 \dots; \xi, \hat{w}_k) \\ &= -\frac{c}{2} \sum_{i,k} (-)^{i+k} \Phi_{ds}(\xi, z_i) \Phi_{ds}(\xi; w_k) \Phi_{ds}(\hat{z}_i; \hat{w}_k) \end{aligned} \quad (77)$$

This expression tells us that when viewed as a function of z_i and w_j ,

$$\langle T(\xi) \prod_i [\psi_+(z_i) \psi_-(w_i)] \rangle = -\frac{c}{2} \mathcal{A}_{z,w} \left(\frac{1}{(z_1 - \xi)^2 (w_1 - \xi)^2} \frac{1}{(z_2 - w_2)^2} \dots \right) \quad (78)$$

is a spin-singlet function and $T(\xi)$ is a spin-singlet operator.

From the relation

$$\begin{aligned} & \sum_i \left(\frac{1}{(\xi - z_i)^2} + \frac{1}{\xi - z_i} \partial_{z_i} + \frac{1}{(\xi - w_i)^2} + \frac{1}{\xi - w_i} \partial_{w_i} \right) \Phi_{ds} \\ &= \sum_{i,k} \left(\frac{1}{(\xi - z_i)^2} + \frac{1}{\xi - z_i} \partial_{z_i} \right) (-)^{i+k} \Phi_{ds}(z_i; w_k) \Phi_{ds}(\hat{z}_i; \hat{w}_k) \\ & \quad + \sum_{i,k} \left(\frac{1}{(\xi - w_k)^2} + \frac{1}{\xi - w_k} \partial_{w_k} \right) (-)^{i+k} \Phi_{ds}(z_i; w_k) \Phi_{ds}(\hat{z}_i; \hat{w}_k) \end{aligned} \quad (79)$$

one can directly check, by comparing with (77), that the Ward identity (2) is indeed satisfied by the Φ_{ds} wave function if $c = -2$.

To check the Virasoro algebra (4) we need to calculate $\langle T(\xi) T(\tilde{\xi}) \prod_i [\psi_+(z_i) \psi_-(w_i)] \rangle$. Using (75) then (74) we can rewrite

$$\begin{aligned} & \Phi_{ds}(\xi_1, \tilde{\xi}_1, z_{1..}; \xi, \tilde{\xi}, w_{1..}) \\ &= \Phi_{ds}(\xi_1, \tilde{\xi}_1; \xi, \tilde{\xi}) \Phi_{ds} + \sum_{i,k} \Phi_{ds}(\xi_1, \tilde{\xi}_1; \xi, w_k) \Phi_{ds}(z_i; \tilde{\xi}) \Phi_{ds}(\hat{z}_i; \hat{w}_k) (-)^{i+k+1} \\ & \quad + \sum_{i,k} \Phi_{ds}(\xi_1, \tilde{\xi}_1; \tilde{\xi}, w_k) \Phi_{ds}(z_i; \xi) \Phi_{ds}(\hat{z}_i; \hat{w}_k) (-)^{i+k} \\ & \quad + \sum_{i' < i} \sum_{k' < k} \Phi_{ds}(\xi_1, \tilde{\xi}_1; w_{k'}, w_k) \Phi_{ds}(z_{i'}, z_i; \xi, \tilde{\xi}) \Phi_{ds}(\hat{z}_{i'}, \hat{z}_i; \hat{w}_{k'}, \hat{w}_k) (-)^{i+i'+k+k'} \end{aligned} \quad (80)$$

Let $\xi_1 = \xi + \delta_1$ and $\tilde{\xi}_1 = \tilde{\xi} + \delta_2$, we find, as $\delta_{1,2} \rightarrow 0$,

$$\Phi_{ds}(\xi_1, \tilde{\xi}_1, z_{1..}; \xi, \tilde{\xi}, w_{1..}) = \frac{A}{\delta_1^2 \delta_2^2} + \frac{B}{\delta_1^2} + \frac{C}{\delta_2^2} + O(1)$$

The singular terms are cancelled by the subtraction in the definition of T (see (45)). The finite term gives

$$\begin{aligned} & \frac{4}{c^2} \langle T(\xi) T(\tilde{\xi}) \prod_i [\psi_+(z_i) \psi_-(w_i)] \rangle \equiv \frac{4}{c^2} F(\xi, \tilde{\xi}; z_{1..}; w_{1..}) \\ &= - \frac{\Phi_{ds}}{(\xi - \tilde{\xi})^4} + \sum_{i,k} \frac{(-)^{i+k}}{(\xi - \tilde{\xi})^2} \left(\Phi_{ds}(\tilde{\xi}; w_k) \Phi_{ds}(z_i; \xi) + \Phi_{ds}(\xi; w_k) \Phi_{ds}(z_i; \tilde{\xi}) \right) \Phi_{ds}(\hat{z}_i; \hat{w}_k) \\ & \quad + \sum_{i' < i, k' < k} \Phi_{ds}(\xi, \tilde{\xi}; w_{k'}, w_k) \Phi_{ds}(z_{i'}, z_i; \xi, \tilde{\xi}) \Phi_{ds}(\hat{z}_{i'}, \hat{z}_i; \hat{w}_{k'}, \hat{w}_k) (-)^{i+i'+k+k'} \end{aligned} \quad (81)$$

The above is to be compared with the right hand side of (4):

$$\begin{aligned} & F'(\xi, \tilde{\xi}; z_{1..}; w_{1..}) \\ & \equiv \sum_k \left(\frac{1}{(\xi - z_k)^2} + \frac{1}{\xi - z_k} \partial_{z_k} + \frac{1}{(\xi - w_k)^2} + \frac{1}{\xi - w_k} \partial_{w_k} \right) \langle T(\tilde{\xi}) \prod_i \psi_+(z_i) \psi_-(w_i) \rangle \\ & \quad + \left(\frac{2}{(\xi - \tilde{\xi})^2} + \frac{1}{\xi - \tilde{\xi}} \partial_{\tilde{\xi}} \right) \langle T(\tilde{\xi}) \prod_i \psi_+(z_i) \psi_-(w_i) \rangle + \frac{c/2}{(\xi - \tilde{\xi})^4} \Phi_{ds} \end{aligned} \quad (82)$$

Let us compare the structure of poles in F and F' . With the help of (74) and (75), we find, in the limit $\tilde{\xi} \rightarrow \xi$,

$$F = -\frac{1}{(\xi - \tilde{\xi})^4} \Phi_{ds} + \left(\frac{2}{(\tilde{\xi} - \xi)^2} + \frac{1}{\tilde{\xi} - \xi} \partial_{\xi} \right) \sum_{i,k} (-)^{i+k} \Phi_{ds}(\xi; z_i) \Phi_{ds}(\xi; w_k) \Phi_{ds}(\hat{z}_i; \hat{w}_k) + O(1) \quad (83)$$

and, in the limit $\xi \rightarrow z_m$,

$$F = \left(\frac{1}{(\xi - z_m)^2} + \frac{1}{\xi - z_m} \partial_{z_m} \right) \sum_{i,k} (-)^{i+k} \Phi_{ds}(\tilde{\xi}; z_i) \Phi_{ds}(\tilde{\xi}; w_k) \Phi_{ds}(\hat{z}_i; \hat{w}_k) + O(1) \quad (84)$$

and similar result for $\xi \rightarrow w_m$.

Using (77) one can show that $F - F'$, as a function of ξ , has no singularities if we choose $c = -2$. Thus $F - F'$ is a polynomial of ξ since F and F' are holomorphic functions of ξ . Knowing $F, F' \rightarrow 0$ as $\xi \rightarrow \infty$, we find $F = F'$ and the T in (45) satisfies the Virasoro algebra of $c = -2$ for the wave function Φ_{ds} .

To show Φ_q in (60) satisfies the conformal Ward identity we need to use the following property of Φ_q

$$\Phi_q = \sum_k (-)^{i+k} f_{u_1, u_2}(z_i, w_k) \Phi_q(\hat{z}_i; \hat{w}_k) \quad (85)$$

Applying (85) twice we find

$$\begin{aligned} & \Phi_q(\xi + \delta, z_1, \dots; \xi, w_1, \dots; u_1, u_2) \\ &= f_{u_1, u_2}(\xi + \delta, \xi) \Phi_q + \sum_{i,k} (-)^{1+i+k} f_{u_1, u_2}(\xi + \delta, w_k) f_{u_1, u_2}(\xi, z_i) \Phi_q(\hat{z}_i; \hat{w}_k) \end{aligned} \quad (86)$$

Taking $\delta \rightarrow 0$ and subtracting out the δ^{-2} term we obtain

$$\begin{aligned} & \langle T(\tilde{\xi}) \eta(u_1) \eta(u_2) \prod_i \psi_+(z_i) \psi_-(w_i) \rangle \\ &= \frac{c}{2} \frac{1}{8} \frac{(u_1 - u_2)^2 \Phi_q}{(u_1 - \xi)^2 (u_2 - \xi)^2} - \frac{c}{2} \sum_{i,k} (-)^{i+k} f_{u_1, u_2}(\xi, w_k) f_{u_1, u_2}(\xi, z_i) \Phi_q(\hat{z}_i; \hat{w}_k) \end{aligned} \quad (87)$$

The right hand side of (87) can be shown to be equal to

$$\begin{aligned} & \sum_i \left(\frac{1}{(\xi - z_i)^2} + \frac{1}{\xi - z_i} \partial_{z_i} + \frac{1}{(\xi - w_i)^2} + \frac{1}{\xi - w_i} \partial_{w_i} \right) \Phi_q \\ & + \left(\frac{h_\eta}{(\xi - u_1)^2} + \frac{1}{\xi - u_1} \partial_{u_1} + \frac{h_\eta}{(\xi - u_2)^2} + \frac{1}{\xi - u_2} \partial_{u_2} \right) \Phi_q \end{aligned} \quad (88)$$

if we choose $c = -2$ and $h_\eta = -1/8$. Thus η is a primary field with dimension $-1/8$. In fact (88) can be written as

$$f_{du} \Psi_q + \sum_{i,j} (-)^{i+j} f_{dij} \Phi_q(\hat{z}_i; \hat{w}_j) \quad (89)$$

where

$$f_{du} = (u_1 - u_2)^{2h_\eta} \left(\frac{h_\eta}{(\xi - u_1)^2} + \frac{1}{\xi - u_1} \partial_{u_1} + \frac{h_\eta}{(\xi - u_2)^2} + \frac{1}{\xi - u_2} \partial_{u_2} \right) (u_1 - u_2)^{-2h_\eta} \quad (90)$$

and

$$f_{dij} = \left(\frac{1}{(\xi - z_i)^2} + \frac{1}{\xi - z_i} \partial_{z_i} + \frac{1}{(\xi - w_j)^2} + \frac{1}{\xi - w_j} \partial_{w_j} + \frac{1}{\xi - u_1} \partial_{u_1} + \frac{1}{\xi - u_2} \partial_{u_2} \right) f_{u_1, u_2}(z_i, w_j) \quad (91)$$

For $h_\eta = -1/8$ one can show that

$$f_{du} = -\frac{1}{8} \frac{(u_1 - u_2)^2}{(u_1 - \xi)^2 (u_2 - \xi)^2} \quad (92)$$

and

$$f_{dij} = f_{u_1, u_2}(\xi, w_j) f_{u_1, u_2}(\xi, z_i) \quad (93)$$

which leads to the equality between (87) and (88).

REFERENCES

1. F.D.M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983); B. I. Halperin, *Phys. Rev. Lett.* **52**, 1583 (1984). S. Girvin, *Phys. Rev. B* **29**, 6012 (1984); A.H. MacDonald and D.B. Murray, *Phys. Rev. B* **32**, 2707 (1985); M.P.H. Fisher and D.H. Lee, *Phys. Rev. Lett.* **63**, 903 (1989); J.K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989); *Phys. Rev.* **B41**, 7653 (1991).
2. X.G. Wen, *Int. J. Mod. Phys. B* **2**, 239 (1990); *Phys. Rev.* **B40**, 7387 (1989); X.G. Wen and Q. Niu, *Phys. Rev.* **B41**, 9377 (1990).
3. B. Blok and X.G. Wen, *Phys. Rev.* **B42**, 8133 (1990); *Phys. Rev.* **B42**, 8145 (1990); N. Read, *Phys. Rev. Lett.* **65**, 1502 (1990); J. Fröhlich and A. Zee, *Nucl. Phys.* **B364**, 517 (1991); Z.F. Ezawa and A. Iwazaki, *Phys. Rev.* **B43**, 2637 (1991); X.G. Wen and A. Zee, *Phys. Rev. B* **46**, 2290 (1992).
4. S. M. Girvin and A. H. MacDonald, *Phys. Rev. Lett.* **58**, 1252 (1987); S. C. Zhang, T. H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989); N. Read, *Phys. Rev. Lett.* **62**, 86 (1989); Z.F. Ezawa and A. Iwazaki, *Phys. Rev.* **B43**, 2637 (1991).
5. X.G. Wen, *Phys. Rev. Lett.* **66**, 802 (1991); B. Blok and X.G. Wen, *Nucl. Phys.* **B374**, 615 (1992).
6. E. Witten, *Comm. Math. Phys.* **121**, 351 (1989); J. Fröhlich and C. King, *Comm. Math. Phys.* **126**, 167 (1989);
7. G. Moore and N. Read, *Nucl. Phys.* **B360**, 362 (1991).
8. S. Fubini, *Int. J. Mod. Phys.* **A5**, 3553 (1990); S. Fubini and C.A. Lütken, *Mod. Phys. Lett.* **A6**, 487 (1991).

9. G. Christofano, G. Maiella, R. Musto and F. Nicodemi, *Phys. Lett.* **B262**, 88 (1991); *Mod. Phys. Lett.* **A6**, 1779 (1991); *ibid* **A6**, 2985 (1991); *ibid* **A7**, 2583 (1992).
10. F.D.M. Haldane and E.H. Rezayi, *Phys. Rev. B* **31**, 2529 (1985).
11. D.P. Li, *Int. J. Mod. Phys.* **B7**, 2655 (1993); *ibid* **B7**, 2779, 1993. E. Keski-Vakkuri, and X.G. Wen, preprint MIT-CTP-2197 (hep-th/9303155).
12. X.G. Wen, *Int. J. Mod. Phys.* **B6**, 1711 (1992); J. Fröhlich and T. Kerler, *Nucl. Phys.* **B354**, 369 (1991).
13. X.G. Wen, *Phys. Rev. Lett.* **70**, 355 (1993).
14. “*Conformal Invariance and Applications to Statistical Mechanics*”, ed. C. Itzykson, H. Saleur and J.B. Zuber, World Scientific, 1988.
15. A.A. Belavin, A.M. Polyakov and A.B. Zamolochikov, *Nucl. Phys.* **B241**, 333 (1984).
16. M. Greiter, X.G. Wen and F. Wilczek, *Phys. Rev. Lett.* **66**, 3605 (1991); *Nucl. Phys.* **B374**, 567 (1992).
17. N. Read and E.H. Rezayi, Yale and CSU preprint, May 1993.
18. When there are more than two η operators, the correlation may have several conformal blocks. We limit ourselves to consider only two η operators, so that only one conformal block is involved in the correlation.
19. L. Dixon, J.A. Harvey, C. Vafa, and E. Witten, *Nucl. Phys.* **B261**, 620 (1985); *Nucl. Phys.* **B274**, 285 (1986). L. Dixon, D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B282**, 13 (1987).
20. Our $c = 1$ Z_2 -orbifold model here is the one with radius $R = \infty$. Actually it can be viewed as a R^1/Z_2 current algebra model.
21. R.L. Willet, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, *Phys. Rev. Lett.* **59**, 1776 (1987).
22. F.D.M. Haldane and E.H. Rezayi, *Phys. Rev. Lett.* **60**, 956 (1988); **60**, E1886 (1988).
23. The HR state was first considered from the CFT point of view in Ref. 7, where a CFT of ghosts plus two Gaussian models are invoked. In our treatment, we use a different CFT, which has an explicit global $SU(2)$ symmetry, and allows us to discuss the quantum numbers of the quasiparticles.
24. N. Read, private discussions.