

The Ground State Structure and Modular Transformations of Fractional Quantum Hall States on a Torus

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The structure of ground states of generic FQH states on a torus is studied by using both effective theory and electron wave function. The relation between the effective theory and the wave function becomes transparent when one considers the ground state structure. We find that the non-abelian Berry's phases of the abelian Hall states generated by twisting the mass matrix are identical to the modular transformation matrix for the characters of Gaussian conformal field theory. We also show that the Haldane-Rezayi spin singlet state has a ten fold ground state degeneracy on a torus which indicates such a state is a non-abelian Hall state.

1. Introduction

Recently, fractional quantum Hall (FQH) states were observed in multi-layer two dimensional electron systems [1]. The hierarchical structure of the FQH states in multi-layer systems appears to be different from that of the single-layer systems. This indicates that the FQH states in multi-layer systems may contain new topological orders. Using the Chern-Simons effective theory, it was shown that the possible topological orders in the abelian FQH states are classified by a symmetric integer matrix K [2]. The hierarchical states in the single-layer systems realize only a small subset of the possible topological orders. The multi-layer systems are a natural place to study more general topological orders.

As a definition, a generic (abelian) FQH state with a topological order labeled by K is described by the following effective theory

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} . \quad (0.1)$$

It was proposed that the multi-layer FQH state [3]

$$\Psi_K = \prod_{i < j, I, J} (z_i^{(I)} - z_j^{(J)})^{K_{IJ}} e^{-\frac{1}{4} \sum_{I,i} |z_i^{(I)}|^2} \quad (0.2)$$

is described by the above effective theory. Thus we say that the multi-layer FQH state (2) has a topological order labeled by K . In Eq. (2), $z_{Ii} = x_{Ii} + iy_{Ii}$ are the coordinates of the electrons in the I^{th} layer.

By the statement that Eq. (1) is the low energy effective theory of the FQH state (2) we mean the following. There exists an energy scale E_0 such that the effective theory (1) reproduces all excitations of the FQH state (2) below that energy scale. At first sight, this statement appears to be trivial for the FQH states. This is because the FQH states are incompressible and there is no excitation below the energy gap. When $\det K \neq 0$, (1) also has a finite gap. It seems that (1) describes the low energy excitations (actually no excitation) in (2) even when K in (1) and (2) are different. However this naive picture is incorrect. When we put the FQH states on a compact Riemann surface, the FQH states will have a non-trivial ground state degeneracy (GSD). What is striking is that the GSD depends on the

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topology of the space. In some sense the GSD of the FQH states has non-trivial “dynamics”. In order to say that (1) is the effective theory of the FQH state (2), we have to show that (1) reproduces the correct “dynamics” for the GSD.

We know an universality class describes a class of systems which flow to the same infrared fixed point. For every infrared fixed point we have a so called low energy effective theory to describe the systems at or near the fixed point. The characteristic effective theory of the FQH states, as we will see, is nothing but the Chern-Simons theory.

The low energy effective theory at the infrared fixed point is much simpler than the original high energy theory. The effective theory of the FQH states, containing only finite degrees of freedom at low energies (a consequence of the gap), is the simplest field theory. The field theory with finite number of degrees of freedom is called topological theory and has attracted a lot of attention after Witten’s pioneer work [4].

Another question that we are going to address is how to measure the matrix K in a physical way (say, in numerical calculations and/or in experiments). K as a parameter in the effective theory is not directly measurable. We would like to find, as many as possible, quantum numbers associated with the degenerate ground states, so that by measuring these quantum numbers we can extract information about K and characterize the topological orders in the FQH state. The results obtained in this paper will help us to determine the topological orders from numerical calculations.

Quantum Hall state is closely related to the conformal field theory. The abelian quantum Hall states correspond to Gaussian models defined on a lattice characterized by the same matrix K . Such a connection has appeared in the edge excitations of the Hall states [5]. To further confirm the above connection, we studied the transformation properties of the ground states of the Hall system under modular transformations. We find that the transformation defined by the non-abelian Berry’s phase of Hall states reproduces the modular transformation matrices between the conformal blocks of the Gaussian model.

In section 2, we study the ground states of the effective theory (1) on a torus. We will show that the GSD is equal to $|\det K|$. In section 3 we will study the multi-layer FQH state (2) on a torus and show that the GSD is also given by $|\det K|$. We further show that the global pieces of the ground state wave functions of the effective theory and the FQH states are identical. This indicates that (1) is indeed the effective theory of the FQH states (2) (in the sense discussed above). We also studied the generalized hierarchical states and show in section 4 that (1) can also be the effective theory of the hierarchical states. In section 5 we consider also a special FQH state, the $\nu = \frac{1}{2}$ Haldane-Rezayi state, and we calculate its GSD on a torus. We find that $\text{GSD} = 10$, which cannot be explained by any simple abelian quantum Hall state. This strongly suggests that the Haldane-Rezayi state is a non-abelian Hall state. In section 6 we study other quantum numbers of the ground states, in addition to the GSD and the filling fraction, that provide information about K . The properties of the ground states under translation are studied and new quantum numbers can indeed provide additional information about K . In the case of a 2×2 matrix K , the new quantum numbers completely determine the K matrix. In section 7 we calculate the non-abelian Berry’s phase of the Hall states and discuss its relation to the modular transformations of the conformal blocks. In section 8 we use our results to study the topological orders in some FQH states.

2. The GSD in the Effective Field Theory

In this section we investigate the ground state structure of an effective field theory suggested to describe excitations around the hierarchy FQH states. Our analysis follows the lines of [6]. The action for the effective field theory is

$$S = \int d^3x \left[\frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{4M} g^{\mu\alpha} g^{\nu\beta} f_{I\mu\nu} f_{I\alpha\beta} \right]. \quad (0.3)$$

There are κ $U(1)$ -gauge fields $a_{I\mu}$, $I = 1, \dots, \kappa$ with field strengths $f_{I\mu\nu} = \partial_\mu a_{I\nu} - \partial_\nu a_{I\mu}$ living on a spacetime $\mathbf{R} \times \Sigma$, \mathbf{R} is time, the space Σ is a torus and M is a parameter with a dimension of mass. The coefficients K_{IJ} form a symmetric $\kappa \times \kappa$ -matrix $K = (K_{IJ})$ and all its elements are integers. The metric g is of the form

$$(g^{\mu\nu}) = \begin{pmatrix} g^{00} & 0 \\ 0 & (-g^{ij}) \end{pmatrix} \quad (0.4)$$

and it (together with the parameter M) gives the scale of excitations of the gauge fields.

We will use Weyl gauge $a_{I0} = 0$. Thus we are left with only the spatial parts of the gauge fields: $a_{Ii}(x)$, $i = 1, 2$. On a torus the global and local excitations of the gauge fields can be separated:

$$a_{Ii}(x) = \frac{\theta_{Ii}(x_0)}{L_i} + \tilde{a}_{Ii}(x); \quad (0.5)$$

where L_1 (L_2) is the length of the torus in x_1 (x_2)-direction and each $\tilde{a}_{Ii}(x)$ satisfies

$$\int_{\Sigma} d^2x \tilde{a}_{Ii}(x) = 0 . \quad (0.6)$$

The gauge invariant physical observables are

$$e^{\oint_{C_j} \tilde{a}_I \cdot d\tilde{x}} = e^{i\theta_{Ij}} ; \quad (0.7)$$

where the contour integral is taken around one of the homology cycles C_j , $j = 1, 2$ around the torus. In order of this to be consistent, each θ_{Ij} must be periodic: $\theta_{Ij} + 2\pi = \theta_{Ij}$.

The Weyl gauge condition leads to constraints

$$\begin{aligned} 0 &= \frac{\delta S}{\delta a_{I0}} = \frac{K_{IJ}}{4\pi} (\partial_i a_{Jj} - \partial_j a_{Ji}) \epsilon^{0ij} + \frac{1}{M} g^{ij} \partial_i f_{I0j} \\ &= \frac{K_{IJ}}{4\pi} \epsilon^{0ij} \tilde{f}_{Jij} + \frac{1}{M} g^{ij} \partial_i \tilde{f}_{I0j} \end{aligned} \quad (0.8)$$

Because of the condition (8) the action can now be factorized into

$$\begin{aligned} S &= \int dt \int_{\Sigma} d^2x \left[\frac{K_{IJ}}{4\pi} \frac{\theta_{Ii}}{L_i} \partial_0 \frac{\theta_{Jj}}{L_j} \epsilon^{0ij} + \frac{1}{2} m^{ij} \partial_0 \frac{\theta_{Ii}}{L_i} \partial_0 \frac{\theta_{Ij}}{L_j} \right] \\ &\quad + \int d^3x \tilde{\mathcal{L}}(\tilde{a}_{I\mu}, \partial_{\mu} \tilde{a}_{I\nu}) \\ &= \int dt \left[\frac{K_{IJ}}{4\pi} (\theta_{I2} \dot{\theta}_{J1} - \theta_{I1} \dot{\theta}_{J2}) + \frac{1}{2} m^{ij} \dot{\theta}_{Ii} \dot{\theta}_{Ij} \right] + \tilde{S}_{local} ; \end{aligned} \quad (0.9)$$

where \tilde{S}_{local} is the action for local excitations and the mass matrix is given by the metric and the dimensionful parameter M : $m^{ij} = \frac{1}{M} g^{00} g^{ij}$. We will neglect the local part and concentrate only in the term in the brackets, which is the Lagrangian of the global excitations. (This kind of ‘‘topological’’ Lagrangians have been studied previously in detail *e.g.* in [7] (in planar geometry).)

Let us assume that the mass matrix defined above has an inverse. Then it is easy to move to the Hamiltonian picture. After solving for canonical momenta, Legendre transforming and quantizing we find

$$H = \frac{1}{2} (m^{-1})_{ij} \Sigma_I \left(\frac{\partial}{\partial \theta_{Ii}} - A_{Ii}^{\theta} \right) \left(\frac{\partial}{\partial \theta_{Ij}} - A_{Ij}^{\theta} \right) , \quad (0.10)$$

the Hamiltonian that governs the dynamics of the wavefunction $\psi(\theta_{Ii})$ of the global excitations. Formally it describes a particle moving on a 2κ -dimensional torus parametrised by $(\theta_{I1}, \theta_{I2})$, in a uniform magnetic field produced by the gauge potential A_{Ii}^{θ} . It is convenient to make a change of coordinates such that the mass matrix becomes diagonal. For this we introduce a complex number $\tau = \tau_x + i\tau_y$ so that we can rewrite the mass matrix as

$$m = m_0 \begin{pmatrix} 1 & \tau_x \\ \tau_x & \tau_x^2 + \tau_y^2 \end{pmatrix} . \quad (0.11)$$

Next we diagonalize its inverse $(m^{-1})_{ij}$ with a matrix

$$S = \frac{1}{2\pi} \begin{pmatrix} 1 & \tau_x \\ 0 & \tau_y \end{pmatrix} , \quad (0.12)$$

which we use to define new coordinates (x_I, y_I) :

$$\begin{pmatrix} x_I \\ y_I \end{pmatrix} = S \begin{pmatrix} \theta_{I1} \\ \theta_{I2} \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} 1 & \tau_x \\ 0 & \tau_y \end{pmatrix} \begin{pmatrix} \theta_{I1} \\ \theta_{I2} \end{pmatrix} . \quad (0.13)$$

The periodicity of the torus is now reflected in $(x_I + 1, y_I)$, (x_I, y_I) and $(x_I + \tau_x, y_I + \tau_y)$ being identical points. In the new coordinates the Hamiltonian takes the form

$$H = -\frac{1}{2m_0} \Sigma_I \left[\left(\frac{\partial}{\partial x_I} - iA_{Ix} \right)^2 + \left(\frac{\partial}{\partial y_I} - iA_{Iy} \right)^2 \right] ; \quad (0.14)$$

where, using the Landau gauge, the gauge potentials are

$$(A_{Ix}, A_{Iy}) = \frac{2\pi}{\tau_y} K_{IJ}(-y_J, 0) . \quad (0.15)$$

Now we can proceed to find the general form of the ground state wave function of the Hamiltonian (14). We leave the details in Appendix, where we also discuss the symmetry properties of (14) and identify the translation generators. Here we just state that the general form of the ground state wavefunction is

$$\psi = f(\{z_I\}) e^{-\frac{\pi}{\tau_y} K_{IJ} y_I y_J} , \quad (0.16)$$

where the function $f(\{z_I\})$ is a holomorphic function of complex variables $z_I = x_I + iy_I$. (In terms of the old variables $z_I = \frac{1}{2\pi}(\theta_{I1} + \tau\theta_{I2})$.) In Landau gauge the wavefunction (16) is quasiperiodic:

$$\begin{cases} \psi(x_I + 1) = \psi(x_I) \\ \psi(x_I + \tau_x, y_I + \tau_y) = \psi(x_I, y_I) \exp(-i2\pi K_{IJ} x_J - i\pi\tau_x K_{II}) . \end{cases} \quad (0.17)$$

We use the convention of showing explicitly only the translated arguments of functions. Notice also that we do not sum over the index I . In order to satisfy these conditions, the holomorphic part must obey

$$\begin{cases} f(z_I + 1) = f(z_I) \\ f(z_I + \tau) = f(z_I) \exp(-i\pi\tau K_{II} - i2\pi K_{IJ} z_J) . \end{cases} \quad (0.18)$$

In the special case that K is diagonal, this reduces to the previously studied case of [6]. Now we want to ask what is the most general class of functions that satisfies the conditions above, and in particular, how many of them are linearly independent. In other words, the problem is to find a basis of the space $V(K, \tau)$ of entire functions f of κ complex variables that satisfy the periodicity conditions (18) above.

In ref. [9] p. 122-125, there is a slightly less general problem for entire functions of several complex variables. Following the arguments there, it is fairly straightforward to generalize the result for our case. One can prove that the general form of a function $f(\vec{z}) \in V(K, \tau)$ is

$$f(\vec{z}) = \sum_{\vec{n}} \chi(\vec{n}) e^{i\pi(K\vec{n})\Omega K^{-1}(K\vec{n}) + i2\pi\vec{n}\cdot\vec{z}} ; \quad (0.19)$$

where the matrix $\Omega = \tau I_{\kappa \times \kappa}$, and $\chi(\vec{n})$ is constant for cosets $\vec{\alpha} + KZ^\kappa$ corresponding to the coset lattice Z^κ/KZ^κ . (The notation KZ^κ means the lattice generated by the column vectors of the matrix K , \vec{z} means the κ -component vector (z_I) .) For instance, choosing $\chi(\vec{n})$ as the characteristic functions $\chi_{\vec{\alpha}}(\vec{n})$ of cosets $\vec{\alpha} + KZ^\kappa$ (*i.e.*, $\chi_{\vec{\alpha}}(\vec{n}) = 1$ if there is a vector $\vec{m} \in Z^\kappa$ such that $\vec{n} = \vec{\alpha} + K\vec{m}$, otherwise $\chi_{\vec{\alpha}}(\vec{n}) = 0$) gives a set of basis vectors of the function space $V(K, \tau)$

$$f(\vec{z}) = f_{\vec{\alpha}}^K(\vec{z} | \tau) = \Theta \left[\begin{matrix} K^{-1}\vec{\alpha} \\ \vec{0} \end{matrix} \right] (K\vec{z} | \tau K) \quad (0.20)$$

labeled by integer quantum numbers $\vec{\alpha} = (\alpha_1, \dots, \alpha_\kappa)$ which live in the coset space Z^κ/KZ^κ . Thus the number of independent basis vectors corresponds to the number of elements in the coset space Z^κ/KZ^κ . Since the unit cells of the lattice KZ^κ have volume $|\det K|$ and the unit cells of the lattice Z^κ have volume 1, we conclude that there are $\mathbf{k} \equiv |\det K|$ linearly independent holomorphic functions $f(\vec{z})$ that satisfy the periodicity conditions (18). Hence the ground state degeneracy in the effective field theory is \mathbf{k} .

In general, we could compactify the space into a Riemann surface Σ_g with higher genus g . Without going into details, we note that using the canonical one cycles A_i, B_i ; $i = 1, \dots, g$ on Σ_g and the respective closed one forms w_i, η_i and expanding the gauge connections A^I in this basis, the Lagrangian (1) would factorize into g copies of a system analyzed above. Thus we conclude that the ground state degeneracy of the theory (1) on a general Riemann surface Σ_g (with the real line as the time coordinate axis) would be \mathbf{k}^g .

3. The Multi-layer Wave Function

We have now seen how the degeneracy of the ground state can be analyzed using the effective field theory. However, if the effective field theory is to describe multi-layer systems, it is important to see if we can reproduce the same result analyzing directly the wave function itself. To construct the wavefunction on a torus, we will follow the lines of Haldane and Rezayi in [10] where they studied the Laughlin wave function on a torus.

Let us consider a multi-layer electron system on a torus parametrized by $0 < \xi < 2\pi$ and $0 < \eta < 2\pi$. First let us consider the one-particle wave function. The wave function satisfies a quasiperiodic boundary condition which can be chosen to be

$$\begin{aligned}\psi(\xi + 2\pi, \eta) &= \psi(\xi, \eta) \\ \psi(\xi, \eta + 2\pi) &= e^{-iN_\phi(\xi + \tau_x\eta) - i\pi N_\phi\tau_x} \psi(\xi, \eta) \quad .\end{aligned}\tag{0.21}$$

Let us also assume that the electrons have the mass matrix given by Eq. (11). The mass matrix can be diagonalized by choosing a new coordinate

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} 1 & \tau_x \\ 0 & \tau_y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}\tag{0.22}$$

In terms of the new coordinates, and under gauge choice

$$(A_x, A_y) = \left(y \frac{2\pi N_\phi}{\tau_y}, 0\right)\tag{0.23}$$

the electron in the first Landau level has the following form of the wave function

$$\psi(\xi, \eta) = e^{-\frac{\pi N_\phi}{\tau_y} y^2} F(z)\tag{0.24}$$

where

$$z = x + iy = \frac{\xi + \tau\eta}{2\pi}\tag{0.25}$$

and $F(z)$ is a holomorphic function of z with no poles. Here N_ϕ is the number of the flux quanta on the torus. The boundary condition for ψ (Eq. (21)) translate into the following boundary condition for $F(z)$:

$$\begin{aligned}F(z + 1) &= F(z) \\ F(z + \tau) &= e^{-i(2z + \tau)\pi N_\phi} F(z) \quad .\end{aligned}\tag{0.26}$$

On a plane, a multilayer FQH state is described by the following type of many-body wavefunction

$$\psi_K(z_i^{(I)}) = [\prod_{I=1}^\kappa \prod_{i < j}^{N_I} (z_i^{(I)} - z_j^{(I)})^{K_{II}}] [\prod_{I < J}^\kappa \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} (z_i^{(I)} - z_j^{(J)})^{K_{IJ}}] e^{-\frac{1}{4} e B \sum_{I,i} |z_i^{(I)}|^2} \quad .\tag{0.27}$$

The index I labels the κ different two-dimensional layers and $z_i^{(I)}$ are the coordinates of the N_I electrons in the I^{th} layer. For more general discussion of the wave function, iterative methods to construct it, and other properties of multilayered systems, see *eg.* [11], [12] and [13].

Our aim is to study this wave function on a torus, *i.e.*, impose periodic boundary conditions and see what restrictions we get to the center of mass part to be added to the wave function.

Let us first for simplicity restrict ourselves to study the two-layer case ($\kappa = 2$). We start with a trial wave function of the general form

$$\psi(z_i, w_i) = F(z_i, w_i) e^{-\frac{\pi}{\tau_y} N_\phi \sum_i y_i^2 - \frac{\pi}{\tau_y} N_\phi \sum_j v_j^2}\tag{0.28}$$

for electrons with mass matrix (11) in the Landau gauge. Here $z_i = x_i + iy_i$ ($w_i = u_i + iv_i$) are the coordinates of the electrons in the layer $I = 1$ ($I = 2$) and $F(z_i, w_i)$ is the holomorphic part of the wave function. We note that all electrons described by (28) are in the first Landau level. Generalizing the boundary conditions (21) to the many-body wave function, we have

$$\begin{aligned}\psi(z_i + 1) &= \psi(z_i) \\ \psi(z_i + \tau) &= \psi(z_i) e^{-2i\pi N_\phi x_i - i\pi N_\phi \tau_x} \\ \psi(w_i + 1) &= \psi(w_i) \\ \psi(w_i + \tau) &= \psi(w_i) e^{-2i\pi N_\phi u_i - i\pi N_\phi \tau_x} \quad ,\end{aligned}\tag{0.29}$$

in agreement with the conventions used in [10] for the single-layer case. From these conditions we derive the following periodicity requirements for the holomorphic part $F(z_i, w_i)$:

$$\begin{aligned}
F(z_i + 1) &= F(z_i) \\
F(z_i + \tau) &= F(z_i)e^{-i\pi N_\phi \tau - i2\pi N_\phi z_i} \\
F(w_i + 1) &= F(w_i) \\
F(w_i + \tau) &= F(w_i)e^{-i\pi N_\phi \tau - i2\pi N_\phi w_i} .
\end{aligned} \tag{0.30}$$

As in [10], to describe the multilayer wave function on a torus, we expect that the holomorphic part will separate into a function of the center-of-mass coordinates of the electrons and a product of odd Jacobi theta functions¹ for the relative coordinates,

$$F(z_i, w_i) = f_c(Z, W) \prod_{i < j}^{N_1} \theta(z_i - z_j)^{m_1} \prod_{i < j}^{N_2} \theta(w_i - w_j)^{m_2} \prod_{i, j=1}^{N_1, N_2} \theta(z_i - w_j)^n . \tag{0.31}$$

Here $Z = \sum_i z_i$, $W = \sum_j w_j$ are the center-of-mass coordinates of the electrons in the different layers and the exponents m_1, m_2, n are related by the magnetic flux, $N_\phi = N_1 m_1 + N_2 n = N_2 m_2 + N_1 n$. The problem is now to see what are the periodicity requirements for the center-of-mass function and what is the most general function that satisfies them. In particular, we want to see what will result as the degeneracy of the wave functions. Using the properties of the theta functions, we find from the theta function part

$$\begin{aligned}
F(z_i + 1) &= f_c(Z + 1) [\dots] (-1)^{(N_1-1)m_1 + N_2 n} \\
F(z_i + \tau) &= f_c(Z + \tau) [\dots] (-1)^{(N_1-1)m_1 + N_2 n} e^{-i\pi[(N_1-1)m_1 + N_2 n]\tau - i2\pi[(N_1 m_1 + N_2 n)z_i - m_1 Z - nW]} \\
F(w_i + 1) &= f_c(W + 1) [\dots] (-1)^{(N_2-1)m_2 + N_1 n} \\
F(w_i + \tau) &= f_c(W + \tau) [\dots] (-1)^{(N_2-1)m_2 + N_1 n} e^{-i\pi[(N_2-1)m_2 + N_1 n]\tau - i2\pi[(N_2 m_2 + N_1 n)w_i - nZ - m_2 W]}
\end{aligned} \tag{0.32}$$

Comparing these with (30) we finally find the periodicity requirements for the center-of-mass part:

$$\begin{aligned}
f_c(Z + 1, W) &= f_c(Z, W) \\
f_c(Z + \tau, W) &= f_c(Z, W) e^{-i\pi m_1 \tau - i2\pi m_1 Z - i2\pi n W} \\
f_c(Z, W + 1) &= f_c(Z, W) \\
f_c(Z, W + \tau) &= f_c(Z, W) e^{-i\pi m_2 \tau - i2\pi n Z - i2\pi m_2 W}
\end{aligned} \tag{0.33}$$

This looks familiar - these are the same conditions as (18) in section 2 ! So we know that the most general entire holomorphic functions are the functions $f_c(Z, W) = f_\alpha^K(Z, W | \tau)$ in the space $V(K, \tau)$ with a matrix

$$K = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \tag{0.34}$$

and the basis functions were given earlier in formula (20). Thus we know that there can be $|\det K| = |m_1 m_2 - n^2|$ linearly independent choices. Thus we have arrived in the same degeneracy as we found in the effective field theory calculation. In addition, the center-of-mass part of the multilayer wave function has *the same form* as the ground state wave functions of the EFT (1).

Generalizing this result to the κ -layer case is now straightforward. We start again with a trial wave function

$$\psi(z_i^{(I)}) = F(z_i^{(I)}) e^{-\frac{\pi}{\tau_y} N_\phi \Sigma_{I,i} (y_i^{(I)})^2} . \tag{0.35}$$

and the with the (quasi)periodicity requirements for it

$$\begin{aligned}
\psi(z_i^{(I)} + 1) &= \psi(z_i^{(I)}) \\
\psi(z_i^{(I)} + \tau) &= \psi(z_i^{(I)}) e^{-i2\pi N_\phi x_i^{(I)} - i\pi N_\phi \tau x} .
\end{aligned} \tag{0.36}$$

These yield conditions

$$\begin{aligned}
F(z_i^{(I)} + 1) &= F(z_i^{(I)}) \\
F(z_i^{(I)} + \tau) &= F(z_i^{(I)}) e^{-i\pi N_\phi \tau - i2\pi N_\phi z_i^{(I)}}
\end{aligned} \tag{0.37}$$

¹We use the notation $\theta(z)$ for the odd theta function $\theta_{\frac{1}{2}, \frac{1}{2}}(z)$ of [9].

for the holomorphic part. On the other hand, replacing the factors $(z_i^{(I)} - z_j^{(J)})$ in the wave function with odd theta functions we can write

$$F(z_i^{(I)}) = f_c(Z^{(I)}) [\prod_{I=1}^{\kappa} \prod_{i < j}^{N_I} \theta(z_i^{(I)} - z_j^{(I)})^{K_{II}}] [\prod_{I < J}^{\kappa} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} \theta(z_i^{(I)} - z_j^{(J)})^{K_{IJ}}] . \quad (0.38)$$

For this the effects of translations are

$$\begin{aligned} F(z_i^{(I)} + 1) &= f_c(Z^{(I)} + 1) [\dots] [\dots] (-1)^{K_{IJ} N_J - K_{II}} \\ F(z_i^{(I)} + \tau) &= f_c(Z^{(I)} + \tau) [\dots] [\dots] (-1)^{K_{IJ} N_J - K_{II}} e^{-i\pi(K_{IJ} N_J - K_{II})\tau - i2\pi(K_{IJ} N_J z_i^{(I)} - K_{IJ} Z^{(J)})} . \end{aligned} \quad (0.39)$$

There are now constraints on N_I , $N_\phi = K_{IJ} N_J$ for all $I = 1, \dots, \kappa$ since the fluxes through all the layers are of equal size. Inserting these to the conditions (39) above and comparing with (37) we find the quasiperiodicity conditions

$$\begin{aligned} f_c(Z^{(I)} + 1) &= f_c(Z^{(I)}) \\ f_c(Z^{(I)} + \tau) &= f_c(Z^{(I)}) e^{-i\pi K_{II} \tau - i2\pi K_{IJ} Z^{(J)}} \end{aligned} \quad (0.40)$$

for the center-of-mass part of the wave function. These are again the same conditions as (18) in section 2. Therefore the wave functions ψ are classified by the linearly independent functions $f_c = f_{\vec{\alpha}}^K$ in the function space $V(K, \tau)$, and the degeneracy of the ground state wave function is \mathbf{k} , the same result as we found in section 2.

4. The Hierarchy FQH Wave function

There are now various different proposals [14], [15] on the market for the ground state wavefunctions in the hierarchy scheme [16] of the FQHE. In this section we will study one of these different wave functions - Read's proposal in [15] - on a torus.

In [15] the hierarchy electron wave function is written in the form

$$\psi(z_i^{(0)}) = \int \prod_{I=1}^{\kappa-1} \prod_{i=1}^{N_I} d^2 z_i^{(I)} \prod_{I=0}^{\kappa-1} \prod_{i < j} (z_i^{(I)} - z_j^{(I)})^{a_I} \prod_{I=0}^{\kappa-1} \prod_{i,j} (z_i^{(I+1)} - z_j^{(I)})^{b_{II+1}} e^{-\frac{1}{4} e B \Sigma_i |z_i^{(0)}|^2} . \quad (0.41)$$

Here the $z_i^{(0)}$ are the positions of N_0 electrons and the integrals are over coordinates of quasiparticles at levels $I = 1, \dots, \kappa - 1$ in the hierarchy, each level I has N_I quasiparticles at positions $z_i^{(I)}$. The exponents are a_0 (odd, > 0), a_I (even, > 0), $b_{II+1} = \pm 1$ and $b_{\kappa-1\kappa} = 0$.

First of all, if we ignore the integrations and look at the function in the integrand we notice that it can be written in the form (up to an irrelevant overall sign)

$$\psi(z_i^{(I)}) = [\prod_{I=0}^{\kappa-1} \prod_{i < j}^{N_I} (z_i^{(I)} - z_j^{(I)})^{K_{II}}] [\prod_{I < J}^{\kappa-1} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} (z_i^{(I)} - z_j^{(J)})^{K_{IJ}}] e^{-\frac{1}{4} e B \Sigma_i |z_i^{(0)}|^2} , \quad (0.42)$$

where we have used a matrix of coefficients

$$(K_{IJ}) = \begin{pmatrix} a_0 & b_{01} & 0 & \dots & 0 \\ b_{10} & a_1 & b_{12} & & 0 \\ \vdots & b_{21} & \ddots & & \vdots \\ 0 & \dots & 0 & b_{\kappa-1\kappa-2} & a_{\kappa-1} \end{pmatrix} . \quad (0.43)$$

On the other hand, the exponential part can be rewritten as

$$e^{-\frac{\pi}{\tau y} N_\phi \Sigma_i (y_i^{(0)})^2} , \quad (0.44)$$

where $y_i^{(0)}$ is the imaginary part of the electron coordinate $z_i^{(0)} = x_i^{(0)} + iy_i^{(0)}$. Notice that this looks now formally exactly like the multilayer wave function as in previous section, except that only the electron coordinates have an exponential factor. This is related to the fact that the total flux N_ϕ for a homogenous ground state is given by

$$(K_{IJ} N_J) = \begin{pmatrix} N_\phi \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (N_\phi \delta_{I,0}) , \quad (0.45)$$

i.e., only the electrons see a non-zero flux. We can now put the integrand function on a torus as we did before. We start with an expression

$$\psi(z_i^{(I)}) = F(z_i^{(I)}) e^{-\frac{\pi}{\tau_y} N_\phi \Sigma_i (y_i^{(0)})^2} . \quad (0.46)$$

For this we require the periodicity properties

$$\begin{aligned} \psi(z_i^{(I)} + 1) &= \psi(z_i^{(I)}) \\ \psi(z_i^{(I)} + \tau) &= \begin{cases} \psi(z_i^{(0)}) e^{-i2\pi N_\phi x_i^0 - i\pi N_\phi \tau_x} & (I = 0) \\ \psi(z_i^{(I)}) & (I > 0) \end{cases} \end{aligned} \quad (0.47)$$

because of the property (45). Notice that ψ is now actually periodic in quasiparticle coordinates. As before, we write

$$F(Z_i^{(I)}) = f_c(Z^{(I)}) [\prod_{I=0}^{\kappa-1} \prod_{i < j}^{N_I} \theta(z_i^{(I)} - z_j^{(I)})^{K_{II}}] [\prod_{I < J}^{\kappa-1} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} \theta(z_i^{(I)} - z_j^{(J)})^{K_{IJ}}] . \quad (0.48)$$

This has the properties

$$\begin{aligned} F(z_i^{(I)} + 1) &= f_c(Z^{(I)} + 1) [\dots] [\dots] (-1)^{K_{IJ} N_J - K_{II}} \\ F(z_i^{(I)} + \tau) &= f_c(Z^{(I)} + \tau) [\dots] [\dots] (-1)^{K_{IJ} N_J - K_{II}} e^{-i\pi (K_{IJ} N_J - K_{II}) \tau - i2\pi (K_{IJ} N_J z_i^{(I)} - K_{IJ} Z^J)} . \end{aligned} \quad (0.49)$$

Using (45) and comparing (49) with (47) we find that the center-of-mass function has to satisfy the conditions

$$\begin{aligned} f_c(Z^{(I)} + 1) &= f_c(Z^{(I)}) \\ f_c(Z^{(I)} + \tau) &= f_c(Z^{(I)}) e^{-i\pi K_{II} \tau - i2\pi K_{IJ} Z^{(J)}} , \end{aligned} \quad (0.50)$$

i.e., once again we have arrived at the periodicity conditions (18) of section 2. The integrands are classified by the \mathbf{k} linearly indepent functions in the space $V(K, \tau)$. However, to get the *electron* wave function, we must take into account the integrations over the quasiparticle coordinates. Nevertheless we find it very plausible that under certain conditions (*e.g.*, the filling factor $\nu < 1$) these functions are also linearly independent and they have the same \mathbf{k} fold degeneracy on a torus. Thus we can argue that the hierarchy FQH states are also described by the effective field theory (1).

5. The Haldane-Rezayi State

In order to describe a FQHE plateau at $\nu = \frac{5}{2}$ seen in recent experiments [17], Haldane and Rezayi [18] have proposed a state which is a spin singlet and has $\nu = \frac{1}{2}$ (or $\nu = \frac{5}{2}$ including a completely filled Landau level). (See also [19].) The state is of the form

$$\psi_{HR}(z_i, w_i) = \det\left(\frac{1}{(z_i - w_j)^2}\right) \prod_{i < j}^N (z_i - z_j)^2 \prod_{i < j}^N (w_i - w_j)^2 \prod_{i,j} (z_i - w_j)^2 e^{-\frac{1}{4} eB (\Sigma_i (|z_i|^2 + |w_i|^2))} . \quad (0.51)$$

Let us put this state on a torus. We proceed as in the multi-layer case. First we rewrite

$$\psi(z_i, w_i) = F(z_i, w_i) e^{-\frac{\pi}{\tau_y} N_\phi (\Sigma_i y_i^2 + \Sigma_i v_i^2)} , \quad (0.52)$$

where $z_i = x_i + iy_i$, $w_i = u_i + iv_i$. Then we require ψ to be quasiperiodic *i.e.*, it has to satisfy formulas (36). As in the multi-layer case, the periodicity requirements for the holomorphic part are given by (37). Next we rewrite the holomorphic part using theta functions and a separate part for the center-of-mass coordinates,

$$\begin{aligned} F(z_i, w_i) &= f_c(Z, W) \det\left(\frac{\theta_{a,b}(z_i - w_j) \theta_{a',b'}(z_i - w_j)}{\theta_{\frac{1}{2}, \frac{1}{2}}^2(z_i - w_j)}\right) \\ &\prod_{i < j}^N \theta_{\frac{1}{2}, \frac{1}{2}}^2(z_i - z_j) \prod_{i < j}^N \theta_{\frac{1}{2}, \frac{1}{2}}^2(w_i - w_j) \prod_{i,j} \theta_{\frac{1}{2}, \frac{1}{2}}^2(z_i - w_j) . \end{aligned} \quad (0.53)$$

This satisfies

$$\begin{aligned}
F(z_i + 1) &= f_c(Z + 1)[\dots](-1)^{2(a+a')+2(N-1)+2N} \\
F(z_i + \tau) &= f_c(Z + \tau)[\dots](-1)^{2(b+b')+2(N-1)+2N} e^{-i\pi[2(N-1)+2N]\tau - i2\pi[(2N+2N)z_i - 2Z - 2W]} \\
F(w_i + 1) &= f_c(W + 1)[\dots](-1)^{2(a+a')+2(N-1)+2N} \\
F(w_i + \tau) &= f_c(W + \tau)[\dots](-1)^{2(b+b')+2(N-1)+2N} e^{-i\pi[2(N-1)+2N]\tau - i2\pi[(2N+2N)w_i - 2W - 2Z]} .
\end{aligned} \tag{0.54}$$

Combining (54) and (37) and using $N_\phi = 2N + 2N$, we find then the periodicity requirements for the c.o.m part:

$$\begin{aligned}
f_c(Z + 1, W) &= f_c(Z, W)(-1)^{2(a+a')} \\
f_c(Z + \tau, W) &= f_c(Z, W)(-1)^{2(b+b')} e^{-i\pi 2\tau - i2\pi(2Z+2W)} \\
f_c(Z, W + 1) &= f_c(Z, W)(-1)^{2(a+a')} \\
f_c(Z, W + \tau) &= f_c(Z, W)(-1)^{2(b+b')} e^{-i\pi 2\tau - i2\pi(2W+2Z)} .
\end{aligned} \tag{0.55}$$

If we ignore the factors $(-1)^{2(a+a')}$, $(-1)^{2(b+b')}$ for a moment, we notice that this looks again like the conditions (18) in section 2. However, we cannot proceed as before to say that

$$f_c(Z, W) = f_K(Z, W) , \quad K = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} . \tag{0.56}$$

This is because now K is not invertible which is not allowed. Instead we rewrite

$$f_c(Z, W) = \tilde{f}_c(\tilde{Z}, \tilde{W}) , \tag{0.57}$$

where $\tilde{Z} = Z + W$, $\tilde{W} = Z - W$. This function has to satisfy

$$\begin{aligned}
\tilde{f}_c(\tilde{Z} + 1, \tilde{W} \pm 1) &= \tilde{f}_c(\tilde{Z}, \tilde{W})(-1)^{2(a+a')} \\
\tilde{f}_c(\tilde{Z} + \tau, \tilde{W} \pm \tau) &= \tilde{f}_c(\tilde{Z}, \tilde{W}) e^{-i\pi 2\tau - 2i\pi(2\tilde{Z})} (-1)^{2(b+b')} .
\end{aligned} \tag{0.58}$$

We find that the most general solution to this can be factorized as

$$\tilde{f}_c(\tilde{Z}, \tilde{W}) = g(\tilde{Z})h(\tilde{W}) . \tag{0.59}$$

The function $h(\tilde{W})$ is periodic under $\tilde{W} \mapsto \tilde{W} \pm 1$, $\tilde{W} \mapsto \tilde{W} \pm \tau$. Liouville's theorem from complex analysis then tells us that h has to be a constant. The function $g(\tilde{Z})$ depends on the phases $(-1)^{2(a+a')}$, $(-1)^{2(b+b')}$. If they are equal to 1, the function $g(\tilde{Z})$ is (see [9], p. 124)

$$g(\tilde{Z}) = g_\alpha^{++}(\tilde{Z}) \equiv \theta \left[\begin{matrix} \alpha \\ \frac{\alpha}{2} \\ 0 \end{matrix} \right] (2\tilde{Z} \mid 2\tau) , \quad \alpha = 0, 1 . \tag{0.60}$$

Thus we have found that the center-of-mass function depends only on the *combined* c.o.m coordinate $\tilde{Z} = Z + W$ and that there are two possible linearly independent choices for it. The other possible values for the phases $(-1)^{2(a+a')}$, $(-1)^{2(b+b')}$ are 1,-1 ; -1, 1 or both equal to -1, depending on the combination of theta functions in the determinant part of the wave function. The conditions (58) are met by modifying the function $g(\tilde{Z})$ to be

$$\begin{aligned}
g_\alpha^{+-}(\tilde{Z}) &\equiv g_\alpha^{++}(\tilde{Z} + \frac{1}{4}) \\
g_\alpha^{-+}(\tilde{Z}) &\equiv e^{i\pi\tilde{Z}} g_\alpha^{++}(\tilde{Z} + \frac{\tau}{4}) \\
g_\alpha^{--}(\tilde{Z}) &\equiv e^{i\pi(\tilde{Z} + \frac{1}{4})} g_\alpha^{++}(\tilde{Z} + \frac{1}{4}(\tau + 1)) ,
\end{aligned} \tag{0.61}$$

depending on the values of the phases. In each case h is still a constant and there are two possible center-of-mass functions.

Next we need to study the degeneracy arising from rest of the wave function. In the $\det(\dots)$ part of the holomorphic part of the wavefunction we had a product of two theta functions

$$\theta_{a,b}(z_i - w_j) \theta_{a',b'}(z_i - w_j) , \tag{0.62}$$

so we have freedom to choose different combinations of theta functions here. However, there is one subtlety. In principle we could replace (62) by an arbitrary linear combination of products (62) that lead to the same phases $(-1)^{2(a+a')}$, $(-1)^{2(b+b')}$ in (54),(55),(58). Let us list all possible combinations. If both phases are equal to -1, the only combination with this property is $(a, b), (a', b') = (0, \frac{1}{2}), (\frac{1}{2}, 0)$, so there is no problem with linear combinations.

The situation is the same if the phases are 1,-1 when the only possibility is $(a, b), (a', b') = (0, 0), (0, \frac{1}{2})$ or -1,1 when the only possibility is $(0, 0), (\frac{1}{2}, 0)$. But, if the phases both equal to 1, we have four possibilities: $(a, b), (a', b') = (0, 0), (0, 0); (0, \frac{1}{2}), (0, \frac{1}{2}); (\frac{1}{2}, 0), (\frac{1}{2}, 0); (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$. However, some of these products are related by formulas

$$\begin{aligned}\theta_{0,0}^2(z) &= \cos \omega \theta_{0,\frac{1}{2}}^2(z) + \sin \omega \theta_{\frac{1}{2},0}^2(z) \\ \theta_{\frac{1}{2},\frac{1}{2}}^2(z) &= \sin \omega \theta_{0,\frac{1}{2}}^2(z) - \cos \omega \theta_{\frac{1}{2},0}^2(z) , \\ \cos \omega &\equiv \theta_{0,\frac{1}{2}}^2(0)/\theta_{0,0}^2(0) , \quad \sin \omega \equiv \theta_{\frac{1}{2},0}^2(0)/\theta_{0,0}^2(0) .\end{aligned}\tag{0.63}$$

(see [9] p. 23). We could still have an arbitrary linear combination of products $(0, \frac{1}{2}), (0, \frac{1}{2}); (\frac{1}{2}, 0), (\frac{1}{2}, 0)$ in the determinant:

$$\det\left(\frac{c\theta_{0,\frac{1}{2}}^2(z_i - w_j) + d\theta_{\frac{1}{2},0}^2(z_i - w_j)}{\theta_{\frac{1}{2},\frac{1}{2}}^2(z_i - w_j)}\right).\tag{0.64}$$

One may think that this could make the wave function *infinitely* degenerate since c, d could take any real values². However, it is possible to prove the interesting relation³

$$\begin{aligned}&\det\left(\frac{c\theta_{0,\frac{1}{2}}^2(z_i - w_j) + d\theta_{\frac{1}{2},0}^2(z_i - w_j)}{\theta_{\frac{1}{2},\frac{1}{2}}^2(z_i - w_j)}\right) \\ &= r_N(c, d) \det\left(\frac{\theta_{0,\frac{1}{2}}^2(z_i - w_j)}{\theta_{\frac{1}{2},\frac{1}{2}}^2(z_i - w_j)}\right) + s_N(c, d) \det\left(\frac{\theta_{\frac{1}{2},0}^2(z_i - w_j)}{\theta_{\frac{1}{2},\frac{1}{2}}^2(z_i - w_j)}\right),\end{aligned}\tag{0.65}$$

where

$$\begin{aligned}r_N(c, d) &= c(c + d \tan \omega)^{N-1} \\ s_N(c, d) &= d(d + c \cot \omega)^{N-1} .\end{aligned}\tag{0.66}$$

The number N is the same that appears in the formulas (51),(53). This means that every determinant of type (64) can always be written as a linear combination of two independent factors. We have then found that there are only two linearly independent combinations with phases equal to 1, in addition to the three combinations with phases equal to 1,-1; -1,1 and -1,-1. Thus there are five different contributions from the antisymmetric part of the wavefunction and two from the center-of-mass part. Hence the total degeneracy for the Haldane-Rezayi state on a torus is *ten*. This result is in agreement with numerical simulations (ref. [20]). Haldane-Rezayi state does not appear to correspond to any abelian FQH state. For example, an abelian state described by

$$K = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

gives rise to a ten fold degeneracy, but does not give filling fraction $\frac{1}{2}$. (The filling fraction is $\frac{1}{5}$.) Thus the Haldane-Rezayi state is very likely to be a non-abelian state.

6. Representations of Translation Generators and New Quantum Numbers

In this section we study the representations of translation generators in terms of the degenerate ground states. The ground state wave functionals from the EFT were found to be linear combinations of the basis functionals

$$\Psi_{\vec{\alpha}} = f_{\vec{\alpha}}(\vec{z}) e^{-\frac{\pi}{\tau y} K_{IJ} y_I y_J} ;\tag{0.67}$$

²Notice that there is a constraint. If one thinks of (c,d) as points in a plane, there is a cut for the line $c/\sin \omega = -d/\cos \omega = t \in R$. These values make the determinant a constant because of (63).

³This relation holds trivially for $N = 1$. One can then generalize it by an induction argument to $N > 1$ making use of (63), the definition of determinant and elementary row operations.

$$f_{\vec{\alpha}}(\vec{z}) = \Theta \left[\begin{array}{c} K^{-1}\vec{\alpha} \\ \vec{0} \end{array} \right] (K\vec{z} | K\tau) . \quad (0.68)$$

Here $\vec{\alpha}$ labels the cosets $\vec{\alpha} + KZ^\kappa$ and

$$\Theta \left[\begin{array}{c} K^{-1}\vec{\alpha} \\ \vec{0} \end{array} \right] (K\vec{z} | K\tau) = \sum_{\vec{m}} \exp\{i\pi(\vec{m} + K^{-1}\vec{\alpha})\tau K(\vec{m} + K^{-1}\vec{\alpha}) + i2\pi(\vec{m} + K^{-1}\vec{\alpha}) \cdot K\vec{z}\} . \quad (0.69)$$

Let us discuss the effect of magnetic translations combined with gauge transformations on the degenerate ground states. The generators of the transformations act on the ground state wave functions as

$$\begin{aligned} t_{I1}\Psi_{\vec{\alpha}} &= e^{i\varphi_{I1}}\Psi_{\vec{\alpha}}(x_J + K_{IJ}^{-1}, y_J) \\ t_{I2}\Psi_{\vec{\alpha}} &= e^{i\varphi_{I2}} e^{i\pi\tau_x K_{II}^{-1} + i2\pi x_I} \Psi_{\vec{\alpha}}(x_J + \tau_x K_{IJ}^{-1}, y_J + \tau_y K_{IJ}^{-1}) , \end{aligned} \quad (0.70)$$

where $\varphi_{I1}, \varphi_{I2}$ are some arbitrary phases. We give an example of t_{Ii} 's with this property in the Appendix. By evaluating the contributions from the holomorphic part and the exponential part separately and separating out the common factors, we find using the basis (68)

$$\begin{aligned} t_{I1}\Psi_{\vec{\alpha}} &= e^{i\varphi_{I1} + i2\pi K_{IJ}^{-1}\alpha_J} \Psi_{\vec{\alpha}}(\vec{z}) \\ t_{I2}\Psi_{\vec{\alpha}} &= e^{i\varphi_{I2}} \Psi_{\vec{\alpha} + \vec{\delta}_I}(\vec{z}) \end{aligned} \quad (0.71)$$

where the vector $\vec{\delta}_I$ means a vector whose I^{th} component is 1 and others are zero. It will turn out to be useful to replace the labels $\vec{\alpha}$ by new labels $\vec{a} \equiv kK^{-1}\vec{\alpha}$. Using this notation we can rewrite (61) as

$$\begin{aligned} t_{I1}\Psi_{\vec{a}} &= e^{i\varphi_{I1} + i2\pi\frac{1}{k}a_i} \Psi_{\vec{a}}(\vec{z}) \\ t_{I2}\Psi_{\vec{a}} &= e^{i\varphi_{I2}} \Psi_{\vec{a} + \vec{\Delta}_I}(\vec{z}) \end{aligned} \quad (0.72)$$

where the vector $\vec{\Delta}_I$ means the I^{th} column vector of the matrix kK^{-1} , $k \equiv \det K$.

Let us make two brief comments. Since it is arbitrary which direction in the torus we call the direction for t_{I1} translation and which for t_{I2} translation, this symmetry between t_{I1}, t_{I2} must manifest itself somehow. We could have defined a different basis from (68) by taking $\chi_{\vec{\alpha}}(\vec{n}) = \exp(2\pi i\vec{n} \cdot (K^{-1}\vec{\alpha}))$ in (19) instead. It is then easy to check that in this basis the t_{I1} translation and the t_{I2} translation operate in the opposite way than above. Thus the symmetry between these two directions manifests itself in the freedom in choosing the basis for the ground state wavefunctions.

We could also study the effect of the combined translations $T_1 \equiv \prod t_{I1}$ and $T_2 \equiv \prod t_{I2}$ (see Appendix). In basis (68) we find

$$\begin{aligned} T_1\Psi_{\vec{a}} &= e^{i\varphi_1 + i2\pi\frac{1}{k}\sum_I a_I} \Psi_{\vec{a}} \\ T_2\Psi_{\vec{a}} &= e^{i\varphi_2} \Psi_{\vec{a} + \vec{\Delta}} , \end{aligned} \quad (0.73)$$

where $\vec{\Delta} = \sum_I \vec{\Delta}_I$. (In the other basis described above T_1 and T_2 would again trade meanings.) We will use these results to try to extract new information about the matrix K and to find new measurable quantum numbers that help us to classify different (abelian) FQH states.

We begin the search for new quantum numbers by looking at the relative phases of the T_1 -quantum numbers. Above we have seen that the T_1 -quantum number for a ground state labelled by a vector \vec{a} was $e^{i\varphi_1 + i2\pi\frac{1}{k}\sum_I a_i}$. Thus these quantum numbers for different ground states differ only by a relative phase factor $e^{i2\pi\frac{1}{k}(\sum_I a'_i - \sum_I a_i)}$. We will prove that (at least when K is a 2×2 -matrix) this relative phase factor is always a multiple of a certain factor $e^{i2\pi\frac{1}{k}\phi}$. More importantly, we will show that the number ϕ can, at least in principle, be measured by comparing the relative phases of the T_1 quantum numbers. We will then discuss how this new quantum number can be used in trying to find the K matrix from the observable quantum numbers.

First we need to know how to find all different labels \vec{a} for the $k = \det K$ different ground states. We notice that all labels \vec{a} can be given as linear combinations

$$\vec{a}(c_1, \dots, c_\kappa) = c_1\vec{\Delta}_1 + \dots + c_\kappa\vec{\Delta}_\kappa , \quad (0.74)$$

where $\vec{\Delta}_I$ is the I^{th} column vector of the matrix kK^{-1} . This can be seen in the following way. Let $\vec{\alpha}$ label the k cosets $\vec{\alpha} + KZ^\kappa$. Then the corresponding labels \vec{a} are

$$\vec{a}(\alpha_1, \dots, \alpha_\kappa) = kK^{-1}\vec{\alpha} = \alpha_1\vec{\Delta}_1 + \dots + \alpha_\kappa\vec{\Delta}_\kappa . \quad (0.75)$$

If we then take $\vec{a}(c_1, \dots, c_\kappa)$ as in (74) and let the c_I run over all integers, we just get the same labels (75) again.

Next we notice that since all labels \vec{a} are of the form (74), the \vec{a} -dependent part in the T_1 -quantum number (73) is of the form

$$a \equiv \sum_I a_I = c_1 \phi_1 + \dots + c_\kappa \phi_\kappa \pmod{k}, \quad (0.76)$$

where $\phi_I = \sum_J (\Delta_I)_J$ (*i.e.*, sum of the entries in the I^{th} column in kK^{-1}). Temporarily, let us not worry about the modding out by k . The equation (76) is a linear Diophantine equation for c_1, \dots, c_κ with integer coefficients $\phi_1, \dots, \phi_\kappa$. Thus we can use the following well-known theorem (see eg. [21] p. 30): there are integer solutions c_1, \dots, c_κ to (76) if and only if the greatest common divisor of the ϕ_I 's, $\phi \equiv \gcd(\phi_1, \dots, \phi_\kappa)$, is a divisor of a . This means that if the c_I in Eqn. (76) are taken to be integers, a will always be a multiple of ϕ . Conversely, for any multiple $a = n\phi$, $n = 0, 1, 2, \dots$ there is an integer solution for the c_I . But, it might be that the modding out by k destroys this picture. However, at least in the case of $\kappa = 2$ it is very easy to prove that also k is a multiple of ϕ , so modding out does not confuse the above result. Thus we know that there is at least one ground state $\psi_{\vec{a}}$ with $a = n\phi$ for every $n = 0, 1, \dots, \frac{k}{\phi}$. This means that by studying the relative phases of the T_1 -quantum numbers we notice that they change by factor $e^{i2\pi \frac{1}{k} \phi}$. If we also know the GSD = k we may then⁴ find ϕ .

We have now *three* quantum numbers that depend on the matrix K : the filling fraction ν , the ground state degeneracy $k = \det K$ and the ‘‘relative phase factor’’ ϕ . There is however one consistency check to be made. Actually the matrix K is defined only up to an equivalence transform $K' = W^T K W$, where $W \in SL(\kappa, Z)$ and $(1, 1, \dots, 1)W = (1, 1, \dots, 1)$. The filling factor and the GSD are independent of these equivalent transformations. We need to ensure that ϕ is also independent. We do this for a 2×2 -matrix.

Let us write the K matrix as

$$K = \begin{pmatrix} A & B \\ B & C \end{pmatrix}. \quad (0.77)$$

Solving the conditions for W we find the equivalent matrices K' parametrized by an integer n :

$$K' = \begin{pmatrix} A + n^2 k \nu + 2n(A - B) & B + n^2 k \nu + n(A - C) \\ B + n^2 k \nu + n(A - C) & C + n^2 k \nu + 2n(B - C) \end{pmatrix}. \quad (0.78)$$

For K (77) the quantum number $\phi = \gcd(\phi_1, \phi_2) = \gcd(C - B, A - B)$. We can rewrite A and C as

$$A = r_1 \phi + B, \quad C = r_2 \phi + B \quad (0.79)$$

with some integer factors r_1, r_2 . We solve for the sum $r_1 + r_2$ using

$$k\nu = A - 2B + C = (r_1 + r_2)\phi \Leftrightarrow r_1 + r_2 = \frac{k\nu}{\phi}. \quad (0.80)$$

Since $\frac{k\nu}{\phi}$ is an integer, ϕ is a divisor of $k\nu$. Therefore ϕ is invariant under the equivalence transformations (78):

$$\gcd(A - B + nk\nu, C - B - nk\nu) = \gcd(A - B, C - B). \quad (0.81)$$

The matrix K (77) depends on three parameters. Can we therefore recover it from the three quantum numbers k, ν and ϕ ? To solve K we would need to find r_1, r_2 separately. This we cannot do directly. However we can use

$$k = AC - B^2 \Leftrightarrow -\phi r_1^2 + (r_1 + r_2)A = \frac{k}{\phi}. \quad (0.82)$$

Using (80) we can now think of (82) as a linear Diophantine equation

$$-\phi r_1^2 + \frac{k\nu}{\phi} A = \frac{k}{\phi} \quad (0.83)$$

⁴We need this assumption since usually we may find several candidates for ϕ if we try to guess it from phases $e^{i2\pi \frac{1}{k} \phi}$. However, if we also know what k is, we can choose the right ϕ since we know that it has to be a divisor of k .

with integer coefficients and r_1^2 , A as unknowns. To be able to describe the system with a 2×2 matrix K we first need to find integer solutions r_1^2, A to (83). This is impossible if $d \equiv \gcd(-\phi, \frac{k\nu}{\phi})$ is not a divisor of $\frac{k}{\phi}$. If it is, we can use the well known algorithm for solving the equation (83) based on the Euclidian algorithm for finding the greatest common divisor of $\phi, \frac{k\nu}{\phi}$. (The algorithm is described *e.g.* in [22].) When we find a solution r_{10}^2, A_0 of (83), all the other solutions are given by

$$r_1^2 = r_{10}^2 + t \frac{k\nu}{d\phi}, \quad A = A_0 + t \frac{\phi}{d}; \quad (0.84)$$

where t is an arbitrary integer parameter. However, r_1^2 has to be a square of an integer. If there are solutions like that, then we can find integer solutions for A, B, C and the system can be described by a 2×2 -matrix K . The parameter t is also related to the fact that K was defined up to an equivalence transformation. It is easy to see that if r_{10}^2, A_0 is a solution that leads to integers A, B, C , by choosing $t = 2n\sqrt{r_{10}^2} + n^2 \frac{k\nu}{\phi}$ in (84) yields the equivalent matrices K' as in (78).

We have thus found that the new quantum number ϕ in addition to the filling fraction and the GSD enables us to *completely classify the abelian FQH states* if they can be described by a 2×2 matrix. In particular, since we found in sections 2 and 3 that the global piece of the multi-layer wavefunction is the same as the one found in the EFT, we can argue that *the second level (abelian) FQH states* (including many double-layer FQH states) might also be *completely classified* by using the method outlined above.

7. Berry's phase

One way to obtain more information about K is to measure the non-abelian Berry's phase associated with the deformation of the electron mass matrix. Let H_τ be the Hamiltonian of the electrons on a torus with mass matrix given by Eq. (11). Assume that at proper filling fraction the electrons described by H_τ form a FQH state labeled by K . Then for each τ , H_τ has $\mathbf{k} \equiv |\det K|$ fold degenerate ground states $|\Phi_n(\tau)\rangle$, $n = 1, \dots, \mathbf{k}$ and $|\Phi_n(\tau)\rangle$ are normalized. We notice that H_τ and $H_{\tau+1}$ actually describe the same system, because $m^{-1}(\tau)$ and $m^{-1}(\tau+1)$ are related by a coordinate transformation $(\xi, \eta) \rightarrow (\xi - \eta, \eta)$. Similarly one can show that H_τ and $H_{-1/\tau}$ describe the same system due to the transformation $(\xi, \eta) \rightarrow (\eta, -\xi)$. Therefore $|\Phi_n(\tau)\rangle$, $|\Phi_n(\tau+1)\rangle$ and $|\Phi_n(-1/\tau)\rangle$ span the same Hilbert space and $|\Phi_n(\tau)\rangle$, $|\Phi_n(\tau+1)\rangle$, and $|\Phi_n(-1/\tau)\rangle$ are related by unitary transformations.

The non-abelian Berry's phase is an unitary matrix that is associated with an adiabatic deformation of the Hamiltonian H_τ [23]. The deformation starts and ends with the same Hamiltonian. Let us denote the deformation path by $\tau(t)|_{t=0}^1$. Then the matrix of the non-Abelian Berry's phase is given by

$$W[\tau(t)] = \mathcal{P} \exp[-i \int_0^1 A(t) dt] W' \quad (0.85)$$

where \mathcal{P} denotes the path ordered product, A is a matrix defined by

$$A_{nm}(t) = i \langle \Phi_n[\tau(t)] | \frac{d}{dt} | \Phi_m[\tau(t)] \rangle \quad (0.86)$$

and W' is the unitary matrix given by

$$W'_{nm} = \langle \Phi_n[\tau(1)] | \Phi_m[\tau(0)] \rangle \quad (0.87)$$

Before going into detailed calculations, let us summarize some general results. The non-abelian Berry's phases induced by the FQH states have the following special properties. For the path that starts and ends with the same τ (*i.e.*, $\tau(0) = \tau(1)$) the non-abelian Berry's phase (denoted as $W(\tau, \tau)$) is a pure phase:

$$W(\tau, \tau)_{mn} = e^{i\theta} \delta_{mn} \quad (0.88)$$

The value of θ may differ from path to path. If the path connects τ and $\tau+2$ (*i.e.*, $\tau(1) = \tau(0) + 2$) then the corresponding non-Abelian Berry's phase (denoted as $W(\tau, \tau+2)$) is longer a pure phase. However only the phase of $W(\tau, \tau+2)$ depends on different choices of paths connecting τ and $\tau+2$. Thus $W(\tau, \tau+2)$ can be written in the following form

$$W(\tau, \tau+2) = e^{i\theta} U \quad (0.89)$$

where $U \in SU(\mathbf{k})/Z_{\mathbf{k}}$ is independent of the paths. Similarly the non-Abelian Berry's phase $W(\tau, -1/\tau)$ associated with paths connecting τ and $-1/\tau$ has a form

$$W(\tau, -1/\tau) = e^{i\theta} S \quad (0.90)$$

Again $S \in SU(\mathbf{k})/Z_{\mathbf{k}}$ is independent of the paths.

Because the two matrices U and S are path independent, they reflect the intrinsic properties of the FQH state. U and S , as $\mathbf{k} \times \mathbf{k}$ matrices, contain a lot of information about the topological orders. In particular they contain the information about the matrix K .

We would like to mention that the two transformations $\tau \rightarrow \tau + 2$ and $\tau \rightarrow -\frac{1}{\tau}$ generate a sub-group Γ_2 of the so called moduli group

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad (0.91)$$

where $a, b, c, d, \in Z$ are integers and $ad - bc = +1$, *i.e.*,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z). \quad (0.92)$$

The pair $\{U, S\}$, being associated with the generators $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of Γ_2 , generates a \mathbf{k} dimensional projective representation of Γ_2 . We see that the topological orders in the FQH states are closely related to the projective representations of the moduli group.

First let us prove that the non-abelian part of the non-abelian Berry's phase is path independent. We will use the multilayer FQH wave function as an example to perform our calculation. The wave function in section 3 was constructed under the gauge $(A_x, A_y) = (y \frac{2\pi N_\phi}{\tau_y}, 0)$. To calculate the Berry's phase it is convenient to choose a different gauge

$$(A_\xi, A_\eta) = (\eta \frac{N_\phi}{2\pi}, 0) \quad (0.93)$$

The new gauge condition is independent of τ . We would like to remind the reader that the torus is parametrized by $0 < \xi < 2\pi$ and $0 < \eta < 2\pi$. The two sets of coordinates (x, y) and (ξ, η) are related through $x + iy \equiv z = \xi + \tau\eta$. Under the new gauge choice the multilayer wave function has a form

$$\psi(z_i^{(I)}) = F(z_i^{(I)}) e^{\frac{i\tau}{4\pi} N_\phi \sum_{I,i} (\eta_i^{(I)})^2} \quad (0.94)$$

From the boundary condition of ψ :

$$\psi(\xi_i^{(I)} + 2\pi\delta_{ij}, \eta_i^{(I)}) = \psi(\xi_i^{(I)}, \eta_i^{(I)}), \quad \psi(\xi_i^{(I)}, \eta_i^{(I)} + 2\pi\delta_{ij}) = e^{-iN_\phi \xi_j^{(I)}} \psi(\xi_i^{(I)}, \eta_i^{(I)}). \quad (0.95)$$

one can show that the $F(z_i^{(I)})$ satisfies exactly the same boundary condition in section 3 (Eq. (37)). Thus, repeating the previous calculation, we find that F is given by (38) with the center-of-mass wave function given by (20). We will denote the electron wave function as $\psi_{\bar{\alpha}}(\xi_i^{(I)}, \eta_i^{(I)}|\tau)$ if the center-of-mass wave function is chosen to be $f_c = f_{\bar{\alpha}}^K$ in Eq. (20). We stress that the wave function $\psi(\xi_i^{(I)}, \eta_i^{(I)}|\tau)$ does not depend on τ^* as one can see from Eq. (38) and (20). This fact is very important for the following discussions.

We first notice that the above degenerate ground state wave functions $\psi_{\bar{\alpha}}(\tau)$ (See Eq. (71)) form an representation of the generalized magnetic translation group. This is because the wave functions of the center-of-mass coordinates are just those discussed in section 2. $\psi_{\bar{\alpha}}(\tau)$ and $\psi_{\bar{\alpha}'}(\tau)$ are orthogonal to each other because they carry different quantum numbers of the commuting unitary magnetic translations t_{I1} . $\psi_{\bar{\alpha}}(\tau)$ and $\psi_{\bar{\alpha}'}(\tau)$ have the same norm since they can be transformed into each other by the unitary magnetic translations t_{I2} . Thus we have

$$\langle \psi_{\bar{\alpha}}(\tau) | \psi_{\bar{\alpha}'}(\tau) \rangle = g(\tau, \tau^*) \delta_{\bar{\alpha}\bar{\alpha}'} \quad (0.96)$$

From (86) and (96) we have

$$\begin{aligned} (\mathcal{A}_\tau)_{\bar{\alpha}\bar{\alpha}'} &= i \int \prod (d\xi_i^{(I)} d\eta_i^{(I)}) \frac{1}{\sqrt{g(\tau, \tau^*)}} \psi_{\bar{\alpha}}^*(\tau) \frac{\partial}{\partial \tau} \left[\frac{1}{\sqrt{g(\tau, \tau^*)}} \psi_{\bar{\alpha}'}(\tau) \right] \\ &= i \sqrt{g(\tau, \tau^*)} \frac{\partial}{\partial \tau} \frac{1}{\sqrt{g(\tau, \tau^*)}} \delta_{\bar{\alpha}\bar{\alpha}'} + i \frac{1}{g(\tau, \tau^*)} \int d^2\theta \psi_{\bar{\alpha}}^* \frac{\partial}{\partial \tau} \psi_{\bar{\alpha}'}. \end{aligned} \quad (0.97)$$

Since $\psi_{\bar{\alpha}}$ is holomorphic in τ , the above can be rewritten as

$$\begin{aligned} (\mathcal{A}_\tau)_{\bar{\alpha}\bar{\alpha}'} &= i \left[-\frac{1}{2} \frac{\partial}{\partial \tau} \ln g(\tau, \tau^*) + \frac{1}{g(\tau, \tau^*)} \frac{\partial}{\partial \tau} g(\tau, \tau^*) \right] \delta_{\bar{\alpha}\bar{\alpha}'} \\ &= i \delta_{\bar{\alpha}\bar{\alpha}'} \frac{1}{2} \frac{\partial}{\partial \tau} \ln g(\tau, \tau^*). \end{aligned} \quad (0.98)$$

Similarly we find that

$$(\mathcal{A}_{\tau^*})_{\bar{\alpha}\bar{\alpha}'} = i \delta_{\bar{\alpha}\bar{\alpha}'} \left(-\frac{1}{2} \right) \frac{\partial}{\partial \tau^*} \ln g(\tau, \tau^*). \quad (0.99)$$

(98) and (99) indicate that the path ordered product in the definition of the non-abelian Berry's phase (85) only contribute to the abelian phase. The non-abelian part of Berry's phase completely comes from the relation between the initial and the final states in Eq. (86).

As we have mentioned, H_τ and $H_{\tau+2}$ describe the same system after a coordinate transformation and a gauge transformation. In the fact if we define a unitary transformation \hat{u} through (for single-particle wave function)

$$\hat{u}\psi(\xi^{(I)}, \eta^{(I)}) = e^{\frac{i}{2\pi}} \sum \eta^{(I)K_{IJ}\eta^{(J)}} \psi(\xi^{(I)} + 2\eta^{(I)}, \eta^{(I)}) \quad (0.100)$$

we can show that $H_{\tau+2} = \hat{u}H_\tau\hat{u}^{-1}$. Thus $g^{-1/2}(\tau + 2, \tau^* + 2)\psi_{\bar{\alpha}}(\tau + 2)$ and $g^{-1/2}(\tau, \tau^*)\hat{u}\psi_{\bar{\alpha}}(\tau)$ are related by a unitary matrix. The matrix can be calculated from the transformation properties the Θ -functions under the modular transformation $\tau \mapsto \tau + 2$. We find that the non-abelian Berry's phase associated with $\tau \mapsto \tau + 2$ is given by

$$U_{\bar{\alpha}\bar{\beta}} = e^{i\phi_U} \delta_{\bar{\alpha}\bar{\beta}} e^{i2\pi\bar{\alpha}K^{-1}\bar{\beta}} \quad (0.101)$$

where ϕ_U is the path dependent $U(1)$ phase. Similarly introducing \hat{s}

$$\hat{s}\psi(\xi^{(I)}, \eta^{(I)}) = e^{-\frac{i}{2\pi}} \sum \xi^{(I)K_{IJ}\eta^{(J)}} \psi(\eta^{(I)}, -\xi^{(I)}) \quad (0.102)$$

we find that $H_{-1/\tau} = \hat{s}H_\tau\hat{s}^{-1}$. The unitary matrix relating $g^{-1/2}(-1/\tau, -1/\tau^*)\psi_{\bar{\alpha}}(-1/\tau)$ and $g^{-1/2}(\tau, \tau^*)\hat{s}\psi_{\bar{\alpha}}(\tau)$ can be again calculated from the modular transformation $\tau \mapsto -1/\tau$ of the Θ -functions. The non-abelian Berry's phase associated with $\tau \mapsto -1/\tau$ is given by

$$S_{\bar{\alpha}\bar{\beta}} = e^{i\phi_S} \frac{1}{\sqrt{\mathbf{k}}} e^{i2\pi\bar{\alpha}K^{-1}\bar{\beta}} \quad (0.103)$$

Up to an overall constant phase, we find that the eigenvalues of U coincide with statistics of the quasiparticles the QH state. Thus the quasiparticle statistics can be determined through the non-abelian Berry's phase without even creating a single quasiparticle.

We would like to point out that the above non-abelian Berry's phase (up to a $U(1)$ phase) is closely related to the modular transformations of the Gaussian model in the conformal field theory. Consider a Gaussian model with κ (real) boson fields ϕ_i :

$$S = \frac{1}{2\pi} \int d^2z \partial_z \phi_i \partial_{\bar{z}} \phi_i, \quad (0.104)$$

where ϕ_i parametrize a κ -dimensional torus, *i.e.*, ϕ_i and $\phi_i + 2\pi R_{ij}l_j$ are identified, l_j are integers and R_{ij} is a real symmetric matrix. The partition function of the above Gaussian model is given by

$$Z = \frac{1}{(\eta(\tau)\eta^*(\tau))^\kappa} \sum_{(p, \bar{p}) \in \Gamma_R} e^{i\pi \sum_i (\tau p_i^2 - \tau^* \bar{p}_i^2)} \quad (0.105)$$

where Γ_R is the lattice

$$\Gamma_R = \left\{ (p_i, \bar{p}_i) = \left(\frac{1}{2}(R^{-1})_{ij}m_j + R_{ij}n_j, \frac{1}{2}(R^{-1})_{ij}m_j - R_{ij}n_j \right); n_j, m_j \in \mathbb{Z}^\kappa \right\}. \quad (0.106)$$

When $(R^{-2})_{ij}$ is an integer matrix K_{ij} with even elements, one can show that the partition function can be written as

$$Z = \frac{1}{(\eta(\tau)\eta^*(\tau))^\kappa} \sum_{\vec{\alpha}} \chi_{\vec{\alpha}}^*(\tau)\chi_{\vec{\alpha}}(\tau) \quad (0.107)$$

where $\vec{\alpha} \in Z^\kappa/KZ^\kappa$ and $\chi_{\vec{\alpha}}$ is given by

$$\chi_{\vec{\alpha}}(\tau) = \sum_{\vec{n} \in Z^\kappa} e^{i\pi(K\vec{n} + \vec{\alpha})\tau K^{-1}(K\vec{n} + \vec{\alpha})}. \quad (0.108)$$

The characters of the Gaussian model are given by $\eta^{-\kappa}(\tau)\chi_{\vec{\alpha}}(\tau)$. We see that the number of the characters (or number of the conformal blocks) is equal to $\mathbf{k} \equiv |\det K|$, the ground state degeneracy of the Hall state on a torus. We also notice that the $\chi_{\vec{\alpha}}(\tau)$ is nothing but the function $f_{\vec{\alpha}}^K(\vec{z} | \tau)$ in Eqn. (20) at $\vec{z} = \vec{0}$. Thus it is not hard to see that, up to a $U(1)$ phase, the modular transformation of the characters is given by the matrices U and S in Eq. (101) and (103) which are the non-abelian Berry's phases of the Hall state. This result illustrates the close relation between the Hall states and the conformal field theory.

8. Examples

In this section we analyze the new quantum numbers for some known FQH states. As a first example we study the abelian $\nu = \frac{1}{2}$ state described by a matrix

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \quad (0.109)$$

In this case the GSD is equal to 8. The vectors $\vec{\alpha}$ labeling the ground states (67) are $\vec{\alpha} = (0, 0); (1, 1); (1, 2); (2, 1); (2, 2); (2, 3); (3, 2); (3, 3)$ and the respective \vec{a} -labels are $\vec{a} = (0, 0); (2, 2); (1, 5); (5, 1); (4, 4); (3, 7); (7, 3); (6, 6)$. The quantum number $\phi = 2$ and the values of a as in (76) are indeed its multiples, $a = 0, 4, 6, 6, 0, 2, 2, 4$ for the above labels. The action of the translation generators (72) is shown in Fig. 1a and the action of the T_2 generator in Fig. 1b. Note that we can block diagonalize the eight-state representation by defining a new basis $(0, 0) \pm (4, 4); (2, 2) \pm (6, 6); (5, 1) \pm (1, 5); (7, 3) \pm (3, 7)$. In this basis the representation decomposes into four blocks of two states (see Fig. 1c). We can then label these basis states using the T_1, T_2^q (q comes from filling factor $\nu = \frac{p}{q}$, here $q = 2$) quantum numbers (since these generators commute). In our example these quantum numbers form a lattice of Fig. 1d and they uniquely label the states. Moreover, the four blocks in the new basis can be labelled by their T_1^2, T_2^2 quantum numbers. If we represent this result in a lattice with T_1^2, T_2^2 as x, y -axes, we find that the four blocks sit in a square (Fig. 1e).

As a second example we look at the $\nu = \frac{1}{3}$ state defined by

$$K = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}. \quad (0.110)$$

This state has GSD=6, the $\phi = 2$, the labels $\vec{\alpha} = (0, 0); (1, 1); (2, 2); (3, 2); (4, 3); (5, 4)$, $\vec{a} = (0, 0); (0, 2); (0, 4); (3, 1); (3, 3); (3, 5)$ and $a = 0, 2, 4, 4, 0, 2$. The representation of (72) is show in Fig. 2a and the representation of (73) in Fig. 2b. The representation separates now in two blocks. In this basis the quantum numbers of T_1, T_2^3 form a lattice of Fig. 2c. The T_1^3, T_2^3 quantum numbers of the blocks are shown in Fig. 2d. Again we find that they sit in a square-like pattern. (It might be that the following is a general feature of the abelian FQH states: if one block diagonalizes the T_1, T_2 representation, the T_1^q, T_2^q quantum numbers for the blocks always form a square-like pattern (with just one point as a special case). We cannot prove this argument yet, however we found after studying several different K -matrices that this always seems to be the case.)

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Note Added

At the completion of this work we received a preprint [24] with some overlap to our results. We have also been informed by Dingping Li that hierarchical wave functions on a torus have also been studied in [25] with overlap to our results.

Appendix

We elaborate on finding the ground state wavefunctions of the Hamiltonian (15),

$$H = -\frac{1}{2m_0}\Sigma_I[(\frac{\partial}{\partial x_I} - iA_{Ix})^2 + (\frac{\partial}{\partial y_I} - iA_{Iy})^2] ; \quad (0.111)$$

where the in the Landau gauge the gauge potentials are

$$(A_{Ix}, A_{Iy}) = \frac{2\pi}{\tau_y}K_{IJ}(-y_J, 0) . \quad (0.112)$$

Le us first change to complex coordinates $z_I = x_I + iy_I$. We define

$$\begin{cases} \partial_{Iz} = \frac{1}{2}(\partial_{Ix} - i\partial_{Iy}) , & A_{Iz} = \frac{1}{2}(A_{Ix} - iA_{Iy}) \\ \partial_{I\bar{z}} = \frac{1}{2}(\partial_{Ix} + i\partial_{Iy}) , & A_{I\bar{z}} = \frac{1}{2}(A_{Ix} + iA_{Iy}) . \end{cases} \quad (0.113)$$

After some algebra we find H in the form

$$H \propto \Sigma_I D_I \bar{D}_I + \text{constant} ; \quad (0.114)$$

where

$$D_I = \partial_{Iz} - iA_{Iz} , \quad \bar{D}_I = \partial_{I\bar{z}} - iA_{I\bar{z}} . \quad (0.115)$$

Now it is easy to find the general form of the ground state wave function. We can take the irrelevant constant to be zero for this purpose. The ground state wave function then satisfies

$$H\psi = \Sigma_I D_I \bar{D}_I \psi = 0 . \quad (0.116)$$

Rewriting the wavefunction in the form $\psi(\{z_I, \bar{z}_I\}) = f(\{z_I\})e^{ig(\{z_I, \bar{z}_I\})}$ where the holomorphic part is written explicitly, we find that the exponent g must be

$$g = i\frac{\pi}{\tau_y}K_{IJ}y_I y_J . \quad (0.117)$$

Thus the ground state wavefunction is of the general form

$$\psi = f(\{z_I\})e^{-\frac{\pi}{\tau_y}K_{IJ}y_I y_J} . \quad (0.118)$$

Let us now turn to examine the symmetry properties of the Hamiltonian. We expect it to be symmetric under translations on a torus spanned by $2N$ complex vectors

$$(z_1, \dots, z_N) = (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1), \\ (\tau, 0, \dots, 0), (0, \tau, 0, \dots, 0), \dots, (0, 0, \dots, 0, \tau) .$$

First we would like to find the magnetic translation generators. The Hamiltonian is

$$H = \frac{1}{2m_0}\Sigma_I(\vec{p}_I \cdot \vec{p}_I - 2\vec{p}_I \cdot \vec{A}_I + \vec{A}_I \cdot \vec{A}_I) . \quad (0.119)$$

Define now

$$\begin{cases} \pi_{Ix} \equiv p_{Ix} - A_{Ix} = p_{Ix} + \frac{2\pi}{\tau_y}K_{IJ}y_J \\ \pi_{Iy} \equiv p_{Iy} - A_{Iy} = p_{Iy} \end{cases} . \quad (0.120)$$

Then a straightforward calculation shows that

$$[\pi_{Ix} - \frac{2\pi}{\tau_y}K_{IJ}y_J, H] = 0 , \quad [\pi_{Iy} + \frac{2\pi}{\tau_y}K_{IJ}x_J, H] = 0 . \quad (0.121)$$

Thus we can construct the following $2N$ translation generators

$$\begin{cases} t_{I1} = \exp [i(K^{-1})_{IJ}(\pi_{Jx} - \frac{2\pi}{\tau_y}K_{JLyL})] = \exp [i(K^{-1})_{IJ}\pi_{Jx} - i\frac{2\pi}{\tau_y}y_I] \\ t_{I2} = \exp [i\tau_x((K^{-1})_{IJ}(\pi_{Jx} - \frac{2\pi}{\tau_y}K_{JLyL})) + i\tau_y((K^{-1})_{IJ}(\pi_{Jy} + \frac{2\pi}{\tau_y}K_{JLxL}))] \end{cases} \quad (0.122)$$

where $I = 1, \dots, N$. These all commute with the Hamiltonian

$$[t_{I1}, H] = [t_{I2}, H] = 0 \quad (0.123)$$

and they satisfy the following ‘‘generalized Heisenberg algebra’’:

$$[t_{I1}, t_{J1}] = 0 = [t_{I2}, t_{J2}] \quad (0.124)$$

for all $I, J = 1, \dots, N$ and

$$t_{I1}t_{J2} = \exp(2\pi i(K^{-1})_{IJ})t_{J2}t_{I1} . \quad (0.125)$$

Notice that we have arrived at the same algebra structure that has been found in the literature of topological Chern-Simons theories [8].⁵

We can also define the following translation generators that commute with the Hamiltonian

$$T_i = \prod_{I=1}^N t_{Ii}, i = 1, 2 \quad (0.126)$$

that we find to satisfy the algebra

$$\begin{aligned} [T_i, T_i] &= 0, i = 1, 2 \\ T_1T_2 &= e^{i2\pi\nu}T_2T_1, \end{aligned} \quad (0.127)$$

where ν is the filling fraction. We study these translation generators and their representations in terms of the ground states in section 6.

⁵See also [24] where a similar result has also been found for Wilson loops.

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Figure captions

Fig. 1a. The action of the translation generators t_{I2} in the basis I for the $K = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ states. Solid line arrow = t_{12} -action, dashed line arrow = t_{22} -action.

Fig. 1b. The action of the T_2 -generator in the basis I.

Fig. 1c. Block diagonalization of the T_2 -representation.

Fig. 1d. The T_1, T_2^2 quantum numbers for the eight states of Fig. 1c. Numbering:

$$\begin{aligned} 1. &= \frac{1}{\sqrt{2}}[(4, 4) - (0, 0)] & 2. &= \frac{1}{\sqrt{2}}[(4, 4) + (0, 0)] \\ 3. &= \frac{1}{\sqrt{2}}[(7, 3) - (3, 7)] & 4. &= \frac{1}{\sqrt{2}}[(7, 3) + (3, 7)] \\ 5. &= \frac{1}{\sqrt{2}}[(2, 2) - (6, 6)] & 6. &= \frac{1}{\sqrt{2}}[(2, 2) + (6, 6)] \\ 7. &= \frac{1}{\sqrt{2}}[(5, 1) - (1, 5)] & 8. &= \frac{1}{\sqrt{2}}[(5, 1) + (1, 5)] \end{aligned}$$

Fig. 1e. The T_1^2, T_2^2 quantum numbers for the four blocks of Fig. 1c.

Fig. 2a. The action of the translation generators t_{I2} in the basis I for the $K = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$ states. Solid line arrow = t_{12} -action, dashed line arrow = t_{22} -action.

Fig. 2b. The action of the T_2 -generator in the basis I.

Fig. 2c. The T_1, T_2^3 quantum numbers for the six states of Fig. 2a. Numbering:

$$\begin{aligned} 1. &= (0, 0); (3, 3) \\ 2. &= (0, 2); (3, 5) \\ 3. &= (0, 4); (3, 1) \end{aligned}$$

Fig. 2d. The T_1^3, T_2^3 quantum numbers for the two blocks of Fig. 2b.

FIGURES