Topological order and edge structure of $\nu = 1/2$ quantum Hall state*

Xiao-Gang Wen

Department of Physics
MIT
77 Massachusetts Avenue
Cambridge, MA 02139

ABSTRACT: We study topological order and edge excitations of $\nu = 1/2$ fractional quantum Hall state (FQH) of spin polarized electrons in single-layer system. We find that the 1/2 FQH state obtained in the numerical study of Greiter *et al.* has a non-abelian topological order suggested by Moore and Read. The edge excitations in such a non-abelian FQH state are found to be described by the $U(1)$ Kac-Moody algebra plus an chiral Majorana fermion theory. The electron and the quasiparticle propagators are calculated. Some experimental consequences are also discussed.

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Spin polarized $\nu = 1/2$ FQH states were recently observed in experiments in double-layer systems.[1,2] The samples in Ref. 2 have a large interlayer hopping rate. The splitting between the symmetric and the anti-symmetric states is of order $\Delta_{SAS} \sim 10K$. Thus the $\nu = 1/2$ FQH state observed in Ref. 2 might correspond to a single-layer FQH state. The numerical calculations in Ref. 3 show that the 1/2 state is possible and likely to appear in single-layer system if the short range repulsion in the Coulomb interaction is reduced. An incompressible $\nu = 1/2$ FQH state was found with an energy gap $\Delta \sim 0.1$ in the thermodynamic limit, if the pseudo potential[4] of the electron interaction is given by $V_1 = \cos(\phi)$, $V_3 = \sin(\phi)$, $\phi = 0.15\pi$. In this paper we are going to study the topological orders[5] in this 1/2 FQH state. We will demonstrate how to use characteristics of finite-size systems to determine the topological orders in the 1/2 state. Our approach should also apply to other FQH states.

Now it is well known that, at a given filling fraction, FQH states may have many different internal structures.[6,5,7,8] The 1/2 FQH state in the V_1-V_3 model is likely to belong to one of the following universality classes:

A) The $\nu = 1/8$ Laughlin state of the charge $2e$ electron pairs.[9,3] The effective theory of this state has a form

$$\mathcal{L} = \sum_{IJ} K_{IJ} \frac{1}{4\pi} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} \tag{1}$$

with K a 1×1 matrix $K = 8$.[8]

B) The second level hierarchical state induced from the quasiparticle *pair* condensate.[10,7,8] The effective theory is given by (1) with $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$.

C) The non-abelian state proposed in Ref. 11 and described by the pfaffian wave function

$$\Psi_{pf}(\{z_i\}) = \mathcal{A}(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \ldots) \prod_{i<j} (z_i - z_j)^2 e^{-\frac{1}{4}\sum_i |z_i|^2} \tag{2}$$

where $\mathcal{A}$ is the antisymmetrization operator.

Other universality classes at $\nu = 1/2$ have more complicated internal structures and are unlikely to appear.. For convenience, we will use $K = 8$, $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$, and $U(1) \times Ising$ to denote the topological orders in the above three universality classes respectively. To determine the topological order in the 1/2 FQH state, we need to measure quantum numbers that can distinguish different topological orders.[5,12,13] One such topological quantum number is the shift $\mathcal{S}$ that appears in the relation $N_\phi = \nu^{-1} N_e - \mathcal{S}$ for FQH states on a sphere. Here N_ϕ is the number of the flux quanta and N_e the number of the electrons. The $K = 8$ and the $U(1) \times Ising$ FQH states have the same shift $\mathcal{S} = 3$,[3] while $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$ state has a shift $\mathcal{S} = 5$.[12] The shift for the 1/2 state in the V_1-V_3 model was found to be $\mathcal{S} = 3$. Thus the topological order in the 1/2 state is not of type $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$.

Before we proceed further to determine whether the topological order in the 1/2 state is of type $K = 8$ or $U(1) \times Ising$, we need to establish that the topological order of type $U(1) \times Ising$ really exists. This requires to find a *local* Hamiltonian that supports an *incompressible* ground state and to show that the ground state really contains a new topological order that differs from $K = 8$.

It was point out that pfaffian wave function Ψ_{pf} is an exact ground state of the following three-body Hamiltonian[3]

$$H_{3bd} = -Uc^\dagger(z_1)c^\dagger(z_2)c^\dagger(z_3)\partial_{z_1^*}\delta(z_1 - z_2)\partial_{z_1}\partial^2_{z_3^*}\delta(z_3 - z_2)\partial^2_{z_3}c(z_3)c(z_2)c(z_1) \qquad (3)$$

The coefficient U is chosen so that the three-electron state $\prod_{i<j}(z_i - z_j)\exp(-\frac{1}{4}\sum_{i=1}^{3}|z_i|^2)$ has unit energy. To show Ψ_{pf} represents an *incompressible* ground state of (3), we calculate, numerically, the energy spectrum of (3) on a sphere with a shift $\mathcal{S} = 3$ (see Fig. 1a). We find a clear energy gap Δ_{sph} for $N_e = 4, 6, 8, 10$. For 10-electron system $\Delta_{sph} \approx 0.6$. It is difficult to estimate the gap in the thermodynamic limit because the gaps for the finite systems do not have a clear scaling behavior. The energy gaps Δ_{pl} obtained from calculations on a plane have a better scaling behavior (See the inset of Fig. 1b). We find $\Delta_{pl} \approx 0.15$ in the thermodynamic limit. The energy gap Δ_{pl} may not be the bulk energy gap Δ_{sph} due to the apparence of the edge. But the former does provide a lower bound for the bulk energy gap: $\Delta_{pl} < \Delta_{sph}$. Naively the algebraic decay of the pairing wave function $\frac{1}{z_1 - z_2}$ as $z_1 - z_2 \to \infty$ seems to suggest the existence of gapless excitation. The above numerical results indicate that the pfaffian wave function is really the *incompressible* ground state of the three-body Hamiltonian (3).

Next we will show that the pfaffian state represents a new universality class different from the $K = 8$ topological order. It was shown in Ref. 13 that FQH states with different topological orders will have different edge excitations. Thus we may use the edge states to probe the bulk topological orders. To study the edge excitations of the pfaffian state, we calculate the spectrum of the Hamiltonian (3) on a plane with 8 electrons in the first 20 orbits (*i.e.,* $0 \le m \le 19$). Here we have chosen the symmetric gauge and m is the angular momentum of single-electron states. The numbers of states (NOS) of the low lying edge excitations at total angular momenta $M_0 + \Delta M$ (where M_0 is the angular momentum of the ground state) are found to be (Fig. 1b)

$$\begin{array}{rccccccccc} \Delta M: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{NOS}: & 1 & 1 & 3 & 5 & 10 & 15 & 25 & 35 & 52 \end{array} \qquad (4)$$

The edge states (the zero-energy states) and the bulk states are clearly separated by a finite gap Δ_{pl}. The numerical results for $N_e = 4, 6, 8$ further indicate that the NOS for $\Delta M \le [\frac{N_e}{2}]$ have reached their thermodynamical value (*i.e.,* the NOS is unchanged as we increase N_e). We know, for a $K = 8$ FQH state, the edge excitations are described by the $U(1)$ Kac-Moody (K-M) algebra (assume N_e is even),[14,15,16,13] which has the following spectrum[17]

$$\begin{array}{rccccccccc} \Delta M: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{NOS}: & 1 & 1 & 2 & 3 & 5 & 7 & 11 & 15 & 22 \end{array} \qquad (5)$$

Comparing (5) and (4) for $\Delta M \le 4$, we conclude that the pfaffian state contains a new topological order that is different from the $K = 8$ topological order. The new topological order is also different from the $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$ topological order. The later contains two branches of edge excitations described by $U(1)\times U(1)$ K-M algebra.[13] The numbers of the edge excitations are given by

$$\begin{array}{rccccccccc} \Delta M: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{NOS}: & 1 & 2 & 5 & 10 & 20 & 36 & 65 & 110 & 180 \end{array} \tag{6}$$

that differ from (4). This way, we established the existence of a new topological order, which was denoted by $U(1)\times Ising$ at the beginning of this paper. Comparing (5), (4) and (6), we notice that the pfaffian state contains more then one branch but less than two branches of edge excitations. The specific heat (per unit length) of the $U(1)\times Ising$ system turns out to be 3/2 times of $\frac{\pi}{6}\frac{T}{v}$ – the specific heat of one branch of edge excitations. In this sense, the pfaffian state contains one and a half branches of edge excitations.

In the following we will show that the edge excitations of the pfaffian state are described by a $U(1)$ K-M algebra plus a chiral Majorana fermion theory. A system of a chiral Majorana fermion is described by the Lagrangian[18]

$$\mathcal{L} = i\lambda(\partial_t - v\partial_x)\lambda \tag{7}$$

where λ is a *real* fermion field: $\lambda^\dagger = \lambda$ and v is the edge velocity. We will put the system on a circle $x \in [0, 2\pi)$. Because λ is real, it can only have two different boundary conditions $\lambda(0,t) = \pm\lambda(2\pi,t)$ (which will be called $\pm$ boundary condition). After quantization, (7) is described by the following Hamiltonian systems:

$$\begin{aligned} &H = v\sum_{k>0}^{+\infty} k a_k a_{-k} \\ &\{a_k, a_{k'}\} = \delta_{k+k'} \\ &\lambda(x) = (2\pi)^{-\frac{1}{2}} \sum_{k=-\infty}^{+\infty} a_k e^{ikx} \end{aligned} \tag{8}$$

where k =integer (or $k=\frac{1}{2}$+integer) for the + (or −) boundary condition. The zero-momentum component a_0 is represented as $a_0 = \frac{1}{\sqrt{2}}(\alpha+\alpha^\dagger)$, where $\{\alpha,\alpha^\dagger\} = 1$. (8) is a free fermion system and is exactly soluble. Because the system (7) has a symmetry $\lambda \to -\lambda$, the quantity $(-)^{N_\lambda}$ is conserved, while the number of the fermions $N_\lambda = \sum_{k\geq 0} a_k a_{-k}$ is not conserved. The theory (7) thus contains four sectors with $\pm$ boundary conditions and even/odd numbers of fermions, which are denoted by (+,even), (−,odd), *etc.* We can only create excitations within the same sector. It is straight forward to work out the numbers of the states at each total momentum k in each sector:

$$\begin{array}{rccccccccccc} \Delta k: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{sector} \\ \text{NOS}: & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 5 & (-,\text{even}) \\ \text{NOS}: & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & (-,\text{odd}) \\ \text{NOS}: & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & (+,\text{odd}) \\ \text{NOS}: & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & (+,\text{even}) \end{array} \tag{9}$$

where $\Delta k = k - k_0$ and k_0 is the momentum of the ground state in each sector. Because the Majorana fermion describe the critical point of the Ising model, we will call the system described by (7) an Ising system.[18]

The system of the $U(1)$ K-M algebra is described by the following Hamiltonian:

$$H = v\nu^{-1}\sum_{k=1}^{+\infty}\rho_k\rho_{-k}$$
$$[\rho_k, \rho_{k'}] = -\nu k\delta_{k+k'} \tag{10}$$

where ν is the filling fraction $\nu = 1/2$, and $\rho(x) = (2\pi)^{-\frac{1}{2}}\sum_k \rho_k e^{ikx}$ is the (1D) charge density on the edge.[14,13] (10) describes a collection of harmonic oscillators and is again exactly soluble. The number of the states at each total momentum k is given by (5). (Note the momentum along the circle Δk is equal to the angular momentum ΔM.) After we put the $U(1)$ K-M algebra and the $Ising$ system together (denoted by $U(1)\times Ising$), we find that the total numbers of the states in each sector at angular momenta ΔM are given by

ΔM :	0	1	2	3	4	5	6	7	8	sector
NOS :	1	1	3	5	10	16	28	43	70	($-$, even)
NOS :	1	2	4	7	13	21	35	55	86	($-$, odd)
NOS :	1	2	4	8	14	24	40	64	100	(+, odd)
NOS :	1	2	4	8	14	24	40	64	100	(+, even)

(11)

Comparing (4) and (11), we see that, for those that have reached the thermodynamical limit (*i.e.,* for $\Delta M \le 4$), the NOS of the edge excitations in the 8-electron pfaffian state exactly match those in the ($-$,even) sector of the $U(1)\times Ising$ system. We also calculate the spectrum of H_{3bd} with 9 electrons in the first 21 orbits. The numbers of the low lying edge excitations are found to be

ΔM :	0	1	2	3	4
NOS :	1	2	4	7	13

(12)

The above NOS are shown to have reached their thermodynamical values and exactly match the ($-$,odd) sector of the $U(1)\times Ising$ system.

From the above discussion, we see that the edge excitations of the pfaffian state are described by the $U(1)\times Ising$ system. This is closely related to the fact that the pfaffian wave function can be constructed from correlation functions in the $U(1)\times Ising$ conformal field theory.[11] The ($-$,even) [($-$,odd)] sector describes the edge excitations for even (odd) numbers of electrons.

Now let us come back to the problem about the topological order in the 1/2 state of the V_1-V_3 model. Consider a deformation of the Hamiltonian of the V_1-V_3 model (denoted by H_{2bd}) to the Hamiltonian of the pfaffian state, H_{3bd}: $H(\lambda) = (1-\lambda)H_{2bd} + \lambda H_{3bd}$. If the ground states of H_{2bd} and H_{3bd} have different topological orders, the energy gap must close for certain λ between 0 and 1. Our numerical calculations for 10-electron systems on sphere (with 17 flux quanta) show that the energy gap is always finite during the deformation. This result indicates that the 1/2 state in the V_1-V_3 model is in the same universality class as the pfaffian state and has the $U(1)\times Ising$ topological order. Thus we expect that the edge excitations in the V_1-V_3 model are also described by the $U(1)\times Ising$ system, because all the FQH states in the same universality class (*i.e.,* with the same topological order) should have the same structure of the edge excitations.[13]

Another way to study the topological orders is to study the ground state degeneracy (GSD) of FQH states on a torus,[5] which is another topological number that characterizes the topological orders in the FQH state. The GSD is six for the pfaffian state,[3] as well as for all the states with the $U(1) \times Ising$ topological order. (Note, in general, the GSD is exact only in the thermodynamic limit.[5]) For the state with $K = 8$ topological order, the GSD is eight, because the state is just the 1/8 Laughlin state. In the inset of Fig. 1a, we present the spectrum of H_{2bd} (10 electrons and $\nu = 1/2$) on a torus (the aspect ratio is chosen to be one), where k_x is the total momentum in the x-direction (in the Landau gauge $A_x = -By$). Due to strong finite size effects, the result is not conclusive. But we can clearly identify the six low lying states. The k_x quantum numbers of those six states agree with the prediction from the pfaffian wave function on torus. This is another evidence that the 1/2 state in the V_1-V_3 model has the $U(1) \times Ising$ topological order.

Another important question is how to measure the topological orders and to distinguish different FQH states with the same filling fraction in real experiments. One way to do so is to measure the tunneling between the edges of the FQH states.[19,13] In general, the tunneling I-V curve is non-linear (at $T = 0$): $I \propto V^{g-1}$ at the peak of the resonant tunneling, or $I \propto V^{2g-1}$ away from the resonance. If the two edges are separated by the vacuum, the tunneling current is due to the electron tunneling and $g = g_e$ is the exponent in the electron propagator: $G_e(t, x = 0) \propto 1/t^{g_e}$. If the two edges are separated by the FQH state, the current is due to the quasiparticle tunneling and $g = g_q$ is the exponent in the quasiparticle propagator: $G_q(t, x = 0) \propto 1/t^{g_q}$. The temperature dependence of the conductance $\sigma = (\frac{dI}{dV})_{V=0}$ is also determined by g: $\sigma \propto T^{g-2}$ at the resonant peak and $\sigma \propto T^{2g-2}$ away from the resonance. g_e and g_q were shown to be topological quantum numbers that independent of details of the electron interaction and the edge potential (provided that all the edge excitations move in the same direction).[13] Thus g_e and g_q are quantum numbers that characterize the topological orders in the bulk states. The value of g_e and g_q were found to be $g_{2e} = 8$ (for electron pairs), $g_q = 1/8$ for the $K = 8$ state and $g_e = 3$, $g_q = 3/8$ for the $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$ and the (331) states.[13] (Note the $K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}$ state and the (331) state in double layer systems have the same structure of edge excitations.) In the following we will calculate g_e and g_q for the $U(1) \times Ising$ state.

The dynamics of the edge excitations is given by (7) and (10), which is a conformal field theory. To calculate the propagator we need to first identify the electron and the quasiparticle operators. The electron operator ψ_e must carry unit electric charge and be a fermionic operator. The only operator that satisfy the above conditions is $\lambda(x) e^{2i\phi(x)}$ which will be identified as the electron operator $\psi_e(x)$. Here $\frac{1}{2\pi}\partial_x \phi = \rho$. From (7) and (8), we find that $\langle \psi_e^\dagger(x,t)\psi_e(0)\rangle \propto (x - vt)^{-3}$ thus $g_e = 3$. This result can be confirmed through a calculation of the electron occupation number n_m in the single particle state $|m\rangle$. Here m is the angular momentum. For the pfaffian state of 10 electrons, we find $n_{m_0} : n_{m_0-1} : ... = 1 : 3.06 : 5.86 : 8.63 : ...$ which agrees with the theoretical prediction for $g_e = 3$:[13] $1 : 3 : 6 : 10 : ....$ Here m_0 is the angular momentum of the last occupied orbit ($m_0 = 17$ for 10 electrons).

Next let us consider the quasiparticle operator. The $U(1) \times Ising$ system contains the following local operators: $e^{i\alpha\phi}$, λ and σ. Here σ is the disorder operator in the $Ising$ system which changes the boundary condition of the fermion λ. Thus σ connects the $+$ sector and the $-$ sector. Not all the above operators are physical, *i.e.,* create an allowed excitation in the electron system. A physical quasiparticle operator must have a single-valued correlation function with the electron operator. This condition is closely related to the single valueness of the electron wave function in presence of the quasiparticle. The condition can be expressed through the operator product expansion: $\psi_e(w_1)\psi_q(w_2) \propto (w_1 - w_2)^\gamma \hat{O}(w_2)$ where we require γ to be an integer. Here w is given by $x + vi\tau$ and τ is the imaginary time $\tau = it$. From the operator product expansion in the $U(1) \times Ising$ system:

$$\begin{aligned} e^{i\alpha\phi(w)}e^{i\beta\phi(0)} &\propto w^{\frac{1}{4}[(\alpha+\beta)^2-\alpha^2-\beta^2]}e^{i(\alpha+\beta)\phi(0)} \\ \lambda(w)\sigma(0) &\propto w^{\frac{1}{2}}\mu(0) \end{aligned} \tag{13}$$

we find that the following operator can be identified as quasiparticle operator: $\psi_q(x) = e^{i\frac{1}{2}\phi(x)}\sigma(x)$. ψ_q carries charge $\frac{1}{4}$ and corresponds to the non-abelian quasiparticle discussed in Ref. 11. From the conformal theory result $\langle\sigma(w)\sigma(0)\rangle \propto w^{1/8}$, we find $\langle\psi_q(x,t)\psi_q(0)\rangle \propto (x - vt)^{-1/4}$ thus $g_q = \frac{1}{4}$.

In the above we have assumed that the $U(1)$ K-M algebra and the $Ising$ system have the same velocity. However this is in general not true. The specific form of the propagators that we wrote down before may not be correct in general. But the result for the exponents should be valid for general situations.

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FIGURE CAPTIONS

Fig. 1 (1a) The spectrum of H_{3bd} on a sphere ($N_e = 10$ and $N_\phi = 17$). L_{tot} is the angular momentum of the $SO(3)$ rotation of the sphere. The inset is the spectrum (first 5 states for each k_x) of H_{2bd} (the V_1-V_3 model) on a torus ($N_e = 10$ and $N_\phi = 20$). k_x and $k_x + N_\phi$ are equivalent. (1b) The spectrum of H_{3bd} on a disk (8 electrons in 20 orbits). The degeneracies of the zero-energy states (*i.e.,* the edge states) at total angular momenta $M = 52, ..., 60$ are 1, 1, 3, 5, 10, 15, 25, 35, 52. The inset is the gap Δ_{pl} for different numbers of electrons.