

**Chiral Operator Product Algebra  
and Edge Excitations of a Fractional Quantum Hall Droplet\***

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ABSTRACT: In this paper we study the spectrum of low-energy edge excitations of a fractional quantum Hall (FQH) droplet. We show how to generate, by conformal field theory (CFT) techniques, the many-electron wave functions for the edge states. And we propose to classify the spectrum of the edge states by the same chiral operator product algebra (OPA) that appears in the CFT description of the ground state in the bulk. This bulk-edge correspondence is suggested particularly for FQH systems that support quasiparticle obeying non-abelian braid statistics, including the  $\nu = 5/2$  Haldane-Rezayi state. Numerical diagonalization to count the low-lying edge states has been done for several non-abelian FQH systems, showing good agreement in all cases with the chiral OPA predictions. The specific heat of the edge excitations in those non-abelian states is also calculated.

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## 1. INTRODUCTION

At low energies the dynamical degrees of freedom of an incompressible fractional quantum Hall (FQH) fluid live on its boundary.<sup>1</sup> They give rise to gapless edge excitations which play important role in many phenomena in FQH systems.<sup>2,3,4,5,6,7</sup> In particular, it has been proposed<sup>8</sup> that the study of edge states may be used to characterize the topological order<sup>9</sup> of an FQH state. (By topological order, we mean “universality class” of properties of an FQH state that reflect certain internal structure of the state and are robust against weak disorders and weak perturbations of electron interactions. Up to now such properties are known to include the quantum numbers (including statistics) of the quasiparticle excitations, the ground state degeneracy on a torus, as well as the spectrum of edge states.)

It has been shown<sup>9,10</sup> that the edge states of abelian FQH states, i.e. those which support quasiparticles obeying abelian braid statistics (such as usual hierarchical FQH states), can be classified by multiple  $U(1)$  current algebras which are characterized by a symmetric  $K$ -matrix with integer elements. This  $U(1)$  current algebra description is known to be closely related to chiral bosons in  $U(1)$  Gaussian conformal field theory (CFT) on the edge<sup>4,6,11,12</sup>. It is remarkable that the edge CFT (in 1+1 dimensional Minkowskian space-time) turns out to be the same as the CFT in the bulk (but in 2-dimensional Euclidean plane) whose correlations give the many-body FQH wave functions for the ground state or quasiparticle excited states.<sup>13,14,11</sup> On the other hand, there may exist FQH states that support quasiparticle excitations obeying non-abelian braid statistics,<sup>13,15,16</sup> among which the  $\nu = 5/2$  Haldane-Rezayi state is an outstanding candidate. How to characterize the spectrum of the edge states of such non-abelian FQH states remains an open question. Some work in this direction has been done in Ref. 17, which suggests that, for the non-abelian Pfaffian state proposed in Ref. 13, the edge excitations are again described by the same CFT that generates the bulk wave functions.

Recently two of us have shown<sup>18</sup> that the many-body wave functions of several (presumably non-abelian) FQH ground states, such as d-wave paired states for spinless or spin-1/2 electrons, admit a CFT description. Namely these FQH wave functions satisfy conformal Ward identities<sup>19,20</sup> and, therefore, can be identified with correlations of a primary field in appropriate CFT. We have explicitly constructed the closed algebra that is generated by the electron operator(s), as primary field(s), through operator product expansions (OPE). This algebra, called by us the center algebra, forms a subalgebra of the complete chiral operator product algebra (OPA) of the CFT. We have been able to extend it to include some disorder operators corresponding to quasiparticles, and thus able to determine the quantum numbers of some of quasiparticle excitations in these FQH systems.

In the same paper it was proposed that the topological order of these FQH states should be characterized by the chiral OPA that appears in their CFT description. It is thus natural to see whether the gapless excitations of these FQH systems on the edge can be described by the same chiral OPA or not. Two questions immediately come to our mind in this regard. First, can we generate the many-body wave functions for the edge excitations as correlations of the same CFT, the way similar to generating those for the bulk ground state and quasiparticle excitations? Second, can we classify the spectrum of edge states by (representations of) the same chiral OPA? This paper reports our research on these two questions.

In Sec. 2 we start with a study of how to generate many-body wave functions that describe edge excitations in the Laughlin  $1/m$ -state by CFT techniques of inserting appropriate screening operators at spatial infinity. We show that the edge states are organized

by representations of the same chiral  $U(1)$  current algebra that appears in the CFT description of bulk states, with the angular momentum of the edge state to be identified with the descendant level in the representation. Then in Sec. 3 we generalize these CFT techniques to several d-wave paired FQH states, especially to the Haldane-Rezayi spin-singlet state which is studied in great detail. In particular, the edge-state spectrum is proposed to be classified by representations of the same chiral OPA appearing in the CFT description for the corresponding bulk states. Thus the number of edge states with a given angular momentum can be predicted from the formulas known in CFT for characters of various chiral OPA. In Sec. 4 we present results of numerical tests of these predictions. Numerical diagonalization is done to count the low-lying edge states for the d-wave paired states for either spinless or spin-1/2 electrons. The results show good agreement in all cases with the predictions from the chiral OPA classification. Also the specific heat of edge excitations in these FQH states is calculated, with an asymptotic analysis of the number of edge states with very large angular momenta.

We would like to stress that to relate the bulk CFT that generate the bulk wave functions to the edge CFT that describes the edge spectrum, we need to introduce a concept of minimal CFT for the bulk CFT. This is because once a FQH wave function can be written as a correlation function in a certain CFT, then the wave function can also be expressed as a correlation in many other CFT which contain the original CFT. The minimal (bulk) CFT for a FQH state not only reproduces the wave function, it is also contained by any other CFT that reproduces the wave function. The bulk CFT constructed in Ref. 18 is automatically the minimal one, because, by construction, the CFT obtained contains the minimal set of primary fields that are needed to construct the wave function. Furthermore the energy-momentum tensor of the minimal CFT can be expressed in terms of the operators that generate the wave function. As we will see in this paper, it is the chiral OPA in this minimal CFT that describes the edge excitations of the corresponding FQH state.

## 2. EDGE EXCITATIONS AND CURRENT ALGEBRA

In this section we are going to study edge excitations of the Laughlin state using a current algebra approach.

We note that the  $\nu = 1/m$  Laughlin state (in the symmetric gauge)

$$\prod_{i < j} (z_i - z_j)^m e^{-\frac{1}{4} \sum_i |z_i|^2} \equiv \Phi_m(z_i) e^{-\frac{1}{4} \sum_i |z_i|^2} \quad (1)$$

is a zero-energy state of a Hamiltonian with the following two-body interaction<sup>21</sup>

$$V(z_1 - z_2) = - \sum_{n=1}^{(m-1)/2} C_n \partial_{z_1}^{2n-1} \delta(z_1 - z_2) \partial_{z_1}^{2n-1} \quad (2)$$

with  $C_n > 0$ . (The Haldane's pseudo-potentials<sup>22</sup> for the above interaction are given by  $V_{2l+1} > 0$  for  $2l + 1 < m$  and  $V_{2l+1} = 0$  for  $2l + 1 \geq m$ .) There are also many other zero-energy states for this Hamiltonian. In fact any antisymmetric wave function of form

$$\tilde{\Phi}_m(z_i) e^{-\frac{1}{4} \sum_i |z_i|^2} \quad (3)$$

has zero energy, if and only if  $\tilde{\Phi}_m$  does not contain zeros of order less than  $m$ , *i.e.*,

$$\tilde{\Phi}_m(z_i) = O((z_1 - z_2)^m) \quad (4)$$

as  $z_1 \rightarrow z_2$ . The wave function (1) describes a circular droplet of the incompressible FQH fluid in its ground state. Other zero-energy states correspond to edge-deformed droplets, *i.e.* excited states of the droplet with edge excitations.<sup>5</sup>

Now the question is how to construct the Hilbert space of these zero-energy edge states. One approach is to use the current algebra techniques in CFT. Let us concentrate on the holomorphic part of the wave function ( $\Phi_m$  or  $\tilde{\Phi}_m$  in (1) and (3)). To construct the edge states we need to construct holomorphic functions that satisfies (4). We notice that the Laughlin-Jastrow function  $\Phi_m$  can be written as a correlation of the vertex operators (or primary fields),  $\psi_e$ , in the Gaussian model as follows:<sup>13,14</sup>

$$\Phi_m = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_N} \langle \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle, \quad (5)$$

where

$$\psi_e(z) = e^{i\sqrt{m}\phi(z)}, \quad \Psi_N(z) = e^{-iN\sqrt{m}\phi(z)}. \quad (6)$$

Here  $N$  is the number of electrons,  $h_N = mN^2/2$  the conformal dimension of  $\Psi_N$ , and the scalar field,  $\phi$ , in the Gaussian model is normalized so that  $\langle \phi(z)\phi(0) \rangle = \ln z$ . The factor  $(z_\infty)^{2h_N}$  is included in (5) so that the limit  $z_\infty \rightarrow \infty$  gives rise to a finite function.<sup>23</sup> Let  $j(z) \equiv \partial\phi$  is the  $U(1)$  current in the Gaussian model. Since  $j$  is a local operator, we find that the correlation

$$\tilde{\Phi}_m \propto \langle \oint dz \alpha(z) j(z) \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle, \quad (7)$$

with appropriately chosen holomorphic function  $\alpha(z)$ , satisfies (4) because of the OPE of  $\psi_e(z)$ :

$$\begin{aligned} \psi_e(z)\psi_e(0) &\propto z^m e^{i2\sqrt{m}\phi(z)} + O(z^{m+1}), \\ j(z)\psi_e(0) &\propto \frac{1}{z}\psi_e(0) + O(1). \end{aligned} \quad (8)$$

The integration contour of  $z$  in (7) encloses all the  $\psi_e(z_i)$  operators but not  $\Psi_N$ . If  $\alpha(z)$  has no poles except at  $z_\infty$ , then (7) is finite for finite  $z_i$ , since  $z_\infty \rightarrow \infty$ . Therefore (7) is an eligible wave function for the edge excitations. Introducing

$$j(z) = \sum_{n=-\infty}^{\infty} \frac{j_n}{(z - z_\infty)^{n+1}} \quad (9)$$

and choosing

$$\alpha(z) = (z - z_\infty)^{-n} \quad (n \geq 0), \quad (10)$$

we find that, by shrinking the contour around  $z_\infty$  and letting  $z_\infty \rightarrow \infty$ , (7) becomes

$$\Phi_m^{(n)} = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_N + 2n} \langle \Psi_N^{(n)}(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle, \quad (11)$$

where

$$\Psi_N^{(n)}(z_\infty) = j_n \Psi_N(z_\infty) . \quad (12)$$

Again the factor  $z_\infty^{2h_N+2n}$  is included in (12) to ensure the existence of a finite limit.<sup>23</sup> One can easily show that  $j_n$  satisfy the  $U(1)$  Kac-Moody (KM) algebra

$$[j_n, j_m] = n\delta_{m+n} . \quad (13)$$

The above discussion can be generalized to the case with several insertions of the current operator. The general statement is the following. The ground state wave function can be written as a correlation of  $\psi_e(z_i)$  with a primary field  $\Psi_N$  inserted at infinity. To generate edge excitations above the ground state, we simply replace the primary field  $\Psi_N$  by its current descendants<sup>24</sup>, which are generally of the form (with  $n_1, n_2, \dots, \geq 0$ )

$$\Psi_N^{(n_1, n_2, \dots)} = (j_{-n_1} j_{-n_2} \dots) \Psi_N . \quad (14)$$

Let us call  $l = \sum_i n_i$  the level of the descendant field  $\Psi_N^{(n_1, n_2, \dots)}$ . The wave function associated with it is given by

$$\Phi_m^{(n_1, n_2, \dots)} = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_N+2l} \langle \Psi_N^{(n_1, n_2, \dots)}(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle . \quad (15)$$

Such wave function has zero energy and can be identified as an edge excitation.

We would like to show that all the edge states generated by the level- $l$  descendant fields have a total angular momentum  $L = M_0 + l$ , where  $M_0 = mN(N-1)/2$  is the total angular momentum of the ground state. We first note that the angular momentum of the ground state can be expressed in terms of conformal dimensions,  $h_e$  and  $h_N$ , of  $\psi_e$  and  $\Psi_N$ . Under a conformal transformation  $z \rightarrow w = f(z)$ , the correlation of primary fields satisfies

$$\langle \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle = (f'(z_\infty))^{h_N} \prod_i (f'(z_i))^{h_e} \langle \Psi_N(w_\infty) \prod_{i=1}^N \psi_e(w_i) \rangle . \quad (16)$$

Choosing  $f(z) = \lambda z$ , we have

$$\langle \Psi_N(\lambda z_\infty) \prod_{i=1}^N \psi_e(\lambda z_i) \rangle = \lambda^{-h_N - N h_e} \langle \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle , \quad (17)$$

which implies that

$$\Phi_m(\lambda z_i) = \lambda^{h_N - N h_e} \Phi_m(z_i) . \quad (18)$$

For  $\lambda = e^{i\theta}$ , this transformation is nothing but a rotation by angle  $\theta$  in the complex plane. Thus the angular momentum of the ground state is

$$M_0 = h_N - N h_e . \quad (19)$$

With  $h_N = mN^2/2$  and  $h_e = m/2$ , we see that  $M_0 = mN(N-1)/2$  as expected for the Laughlin  $1/m$ -state. More generally, since the dimension of a current descendant  $\Psi_N^{(n_1, n_2, \dots)}(z)$  is<sup>24</sup> the sum  $h_N + l$ , where  $l$  is its level, we find that under  $z \rightarrow w = \lambda z$

$$\Psi_N^{(n_1, n_2, \dots)}(z) = \lambda^{h_N + l} \Psi_N^{(n_1, n_2, \dots)}(w) ; \quad (20)$$

Thus, from (15) and (20) we can see that the edge-excited state described by  $\Phi_m^{(n_1, n_2, \dots)}$  carries an angular momentum of  $L = h_N + l - Nh_e = M_0 + l$ . As a general rule, valid for any FQH states that can be generated by CFT, the angular momentum of an edge excitation is equal to that of the ground state plus the level of the descendant level of the associated insertion at infinity.

Note that the descendant fields  $\Phi_m^{(n_1, n_2, \dots)}$  with different  $(n_1, n_2, \dots)$  may not be linearly independent. From the  $U(1)$  KM algebra (13), it is easy to show that the number,  $D_l$ , of linearly independent descendant fields, (14), at level  $l$  is given by the partition number of  $l$ . Mathematically this fact is expressed in terms of the character (of the irreducible representation, with the highest weight state  $\Psi_N$ ) of the  $U(1)$  KM algebra as follows<sup>25</sup>:

$$\text{ch}_N(\xi) \equiv \xi^{h_N} \sum_l D_l \xi^l = \xi^{h_N} \frac{1}{\prod_{n>0} (1 - \xi^n)} . \quad (21)$$

If there is a one-to-one correspondence between the edge states and the descendant fields of  $\Psi_N$ , then we can use the above formula to obtain the number,  $D_L$ , of edge excitations at any given angular momentum  $L$ :

$$\begin{aligned} \text{Ch}_N(\xi) &\equiv \sum_L D_L \xi^L = \text{ch}_N(\xi) \xi^{-Nh_e} \\ &= \frac{1}{\prod_{n>0} (1 - \xi^n)} \xi^{M_0} . \end{aligned} \quad (22)$$

The one-to-one correspondence between the edge states and the descendant fields of  $\Psi_N$  means that different descendant fields always generate linearly independent wave functions of  $z_i$  through (15). Since in (15), the descendant field  $\Psi_N^{(n_1, n_2, \dots)}(z_\infty)$  acts on the state  $\prod_{i=1}^N \psi_e(z_i)$ , we do not expect this correspondence to be true for finite  $N$  and arbitrarily large level  $l = \sum_k n_k$ . On the other hand, it is conceivable that this one-to-one correspondence holds for any finite level  $l$  when  $N$  is very large or in the limit  $N \rightarrow \infty$ . Though we do not know, at present, how to prove this statement within the CFT approach for a generic abelian FQH state, its validity for the  $1/m$ -states can be seen in the following way: It is known<sup>5,6</sup> that the edge states can be generated by multiplying the ground state wave function by symmetric polynomials of electron coordinates  $z_i$ . Mathematically, the number of linearly independent symmetric polynomials of degree  $l$  is precisely given by the partition number of the integer  $l$ . Therefore, there is a close correspondence between the descendant fields in the chiral OPA in the bulk CFT description and the spectrum of the edge states. For Laughlin  $1/m$ -states, the bulk chiral OPA is simply a  $U(1)$  current algebra<sup>13,14,11</sup>; the correspondence between the bulk CFT and the edge spectrum has been known before<sup>17</sup>. In next section, we will show that this relationship can be generalized to non-abelian FQH systems.

### 3. EDGE EXCITATIONS AND OPERATOR PRODUCT ALGEBRA

In the last section we have generated the edge excitations of the Laughlin state by inserting the current operator. One may try to generate edge states by inserting the energy-momentum tensor  $T$ , because the insertions of the energy-momentum tensor also maintain the structure of zeros (4) of the wave functions for the Laughlin state. But one can show that the edge states generated by the energy-momentum tensor is contained in those generated by the current since  $T \propto j^2$ . In the following we will discuss the edge excitations of the Haldane-Rezayi (HR) state, in which case the energy-momentum tensor does generate new edge states.

The HR state is a d-wave-paired spin-singlet FQH state for spin-1/2 electrons. Apart from the usual Gaussian factor, the holomorphic part of the wave function is given by

$$\begin{aligned}\Phi_{HR}(z_i, w_i) &= \Phi_m(z_i, w_i) \Phi_{ds}(z_i, w_i) , \\ \Phi_{ds}(z_i, w_i) &= \mathcal{A}_{z,w} \left( \frac{1}{(z_1 - w_1)^2} \frac{1}{(z_2 - w_2)^2} \dots \right) , \\ \Phi_m(z_i, w_i) &= \left( \prod_{i < j} (z_i - z_j)^m \prod_{i < j} (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^m \right)\end{aligned}\tag{23}$$

which has a filling fraction  $1/m$  with  $m$  an even integer. Here  $z_i$  ( $w_i$ ) are the coordinates of the spin-up (spin-down) electrons, and  $\mathcal{A}_{z,w}$  is an operator which performs separate antisymmetrizations between  $z_i$ 's and between  $w_i$ 's. One can directly check that  $\Phi_{ds}$  is indeed a spin singlet and  $\Phi_m$ , when viewed as an operator, commutes with the total spin operator.

Let us first analyze the structure of zeros of the HR wave function assuming, for simplicity,  $m = 2$ . Let  $z_1 = z_2 + \delta_1$  and  $z_1 = w_1 + \delta_2$ , we find  $\Phi_{HR}$  has the following expansion

$$\begin{aligned}\Phi_{HR} &= \sum_{k=odd} (\delta_1)^k A_k(z_2, \dots; w_1, \dots) ; \\ \Phi_{HR} &= \sum_n (\delta_2)^n B_n(z_2, \dots; w_1, \dots) .\end{aligned}\tag{24}$$

One can directly check that the coefficients

$$A_1 = B_1 = 0 .\tag{25}$$

Therefore the HR state is the exact ground state of the following two-body Hamiltonian

$$H = -V_1 \partial_{z_1^*} \delta(z_1 - z_2) \partial_{z_1} - V_2 \partial_{z_1^*} \delta(z_1 - w_1) \partial_{z_1} ,\tag{26}$$

since  $H$  is positive definite for  $V_i > 0$ , and  $\Phi_{HR}$  has a zero average energy. It has been checked numerically that  $\Phi_{HR}$  is the unique incompressible ground state of  $H$ .<sup>26</sup> Other zero-energy states all have higher angular momenta and are identified as the edge excitations of the ground state.

It was shown in Ref. 18 that the so-called non-abelian part,  $\Phi_{ds}$ , can be written as a correlation of two spin-1/2 primary fields  $\psi_{\pm}$  in a  $c = -2$  CFT. Therefore the ground state

wave function can be written as

$$\Phi_{HR} = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_N} \langle \Psi_N(z_\infty) \prod_{i=1}^{N/2} \psi_{e_+}(z_i) \psi_{e_-}(w_i) \rangle, \quad (27)$$

where

$$\psi_{e_\pm}(z) = \psi_\pm(z) e^{i\sqrt{m}\phi(z)}, \quad (28)$$

$$\Psi_N(z_\infty) = e^{-iN\sqrt{m}\phi(z_\infty)}. \quad (29)$$

Here  $\phi(z)$  is the chiral scalar field in the Gaussian model that reproduces the  $U(1)$  part  $\Phi_m$ . In (27)  $h_N = mN^2/2$  is the dimension of  $\Psi_N$  and  $N$  the total number of spin-up and spin-down electrons. As shown in Ref. 18, the OPE of  $\psi_e$  (for  $m = 2$ )

$$\begin{aligned} \psi_{e_+}(z + \delta_1) \psi_{e_+}(z) &\propto (\delta_1)^3 \psi_1^1(z) e^{2i\sqrt{m}\phi(z)} + O((\delta_1)^4), \\ \psi_{e_+}(z + \delta_2) \psi_{e_-}(z) &\propto e^{2i\sqrt{m}\phi(z)} + O((\delta_2)^2), \end{aligned} \quad (30)$$

guarantees the condition (25) due to the absence of terms linear in  $\delta_1$  and  $\delta_2$ .

Let  $T$  be the energy-momentum tensor of the  $c = -2$  model and  $j$  the  $U(1)$  current of the Gaussian model. The insertion of  $T$  and  $j$  does not affect the local OPE of the  $\psi_e$  operators, and hence the structure of local zeros of the wave function is not affected. Thus we can use both  $T$ - and  $j$ -insertions to generate edge excitations that are zero-energy states of the Hamiltonian (26). Repeating the derivation in the last section, we obtain edge excitations that are generated from the descendant fields of  $\Psi_N$ :

$$\Phi_{HR}^{edge} = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_N + 2l} \langle \Psi_N^{(n_1, \dots, m_1, \dots)}(z_\infty) \prod_{i=1}^{N/2} \psi_{e_+}(z_i) \psi_{e_-}(w_i) \rangle, \quad (31)$$

where

$$l = \sum_i n_i + \sum_k m_k \quad (32)$$

is the level of the descendant field

$$\Psi_N^{(n_1, \dots, m_1, \dots)}(z_\infty) = (L_{-n_1} \dots j_{-m_1} \dots) \Psi_N(z_\infty). \quad (33)$$

Here the fields  $\Psi_N^{(n_1, \dots, m_1, \dots)}(z_\infty)$  are descendants of  $\Psi_N$  generated by the  $U(1)$  current algebra of  $j$  (formed by  $\phi$ ) and the  $c = -2$  Virasoro algebra of  $T$  (formed by  $\psi_\pm$  only).  $L_n$  are the Fourier components of  $T$ :  $T(z) = \sum_{-\infty}^{+\infty} L_n / z^{n+2}$ , which form the  $c = -2$  Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{m+n}. \quad (34)$$

Note both  $T$  and  $j$  are spin singlets. So they generate only the spin-singlet sector of the edge excitations.

The space of the descendant fields is isomorphic to the direct product of the representation of the KM algebra generated by  $j$  and the Verma module of the identity operator generated by  $T$ . This is because  $T$  and  $j$  commute with each other. Similar to the pure  $U(1)$

case in the last section, the edge excitations generated by level- $l$  descendant fields can be shown to have angular momentum  $L = M_0 + l$ , where  $M_0 = h_N - Nh_e = \frac{m}{2}N(N-1) - 2N$  is the angular momentum of the ground state  $\Phi_{HR}$ . Here we have used (19), and  $h_e = \frac{m}{2} + 1$ ,  $h_N = \frac{m}{2}N^2$  (see Ref. 18).

To study the spin- $s$  sector of the edge excitations, we need to study the zero-energy states with total spin  $s$ . The minimum angular momentum state with total spin- $s$  and  $S_z = \sigma$  ( $s = 1/2, 1, 3/2, \dots$ ) is given by

$$\Phi_{HR,s,\sigma} = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_{N,s}} \langle \Psi_{N,s,\sigma}(z_\infty) \prod_{i=1}^{N_+} \psi_{e^+}(z_i) \prod_{i=1}^{N_-} \psi_{e^-}(w_i) \rangle . \quad (35)$$

Here

$$\Psi_{N,s,\sigma}(z_\infty) = \psi_\sigma^s e^{-iN\sqrt{m}\phi(z)} , \quad (36)$$

$h_{N,s}$  is the dimension of  $\Psi_{N,s,\sigma}$ , and  $N_+$  ( $N_-$ ) is the number of spin-up (spin-down) electrons.  $N_\pm$  satisfy  $N_+ - N_- = 2\sigma$  and  $N$  is total numbers of the electrons  $N = N_+ + N_-$ . The state (35) satisfies the condition (25) and is a zero-energy state.  $\psi_\sigma^s$  is the spin- $s$  primary field discussed in Ref. 18, which is generated from the multiple OPE of  $\psi_\pm \equiv \psi_{\pm 1/2}^{1/2}$  and has a conformal dimension

$$h_s = \frac{1}{8}[(4s+1)^2 - 1]. \quad (37)$$

Thus  $h_{N,s} = h_N + h_s = \frac{m}{2}N^2 + \frac{1}{8}[(4s+1)^2 - 1]$ . Other zero-energy states are generated by the descendant fields of  $\Psi_{N,s,\sigma}$ :

$$\Psi_{N,s,\sigma}^{(n_1, \dots; m_1, \dots)}(z_\infty) = (L_{-n_1} \dots j_{-m_1} \dots) \Psi_{N,s,\sigma}(z_\infty) . \quad (38)$$

The wave functions generated by them have the form

$$\Phi_{HR,s,\sigma}^{(n_1, \dots; m_1, \dots)} = \lim_{z_\infty \rightarrow \infty} (z_\infty)^{2h_{N,s} + 2l} \langle \Psi_{N,s,\sigma}^{(n_1, \dots; m_1, \dots)}(z_\infty) \prod_{i=1}^{N_+} \psi_{e^+}(z_i) \prod_{i=1}^{N_-} \psi_{e^-}(w_i) \rangle . \quad (39)$$

They correspond to edge excitations with total angular momentum

$$L = h_{N,s} + l - Nh_e = \frac{m}{2}N(N-1) + h_s - N + l , \quad (40)$$

where  $l$  is the descendant level given by (32).

To study the structure of the space of the descendant fields  $\Psi_{N,s,\sigma}^{(n_1, \dots; m_1, \dots)}$  generated from the primary field  $\Psi_{N,s,\sigma}$ , it is useful to introduce the character associated to the primary field  $\Psi_{N,s,\sigma}$  and its descendants, defined by

$$\text{ch}_{N,s}(\xi) = \sum_n D_n \xi^n , \quad (41)$$

where  $D_n$  is the number of the independent descendant fields of scaling dimension  $n$ . (Recall that the descendant field  $\Psi_{N,s,\sigma}^{(n_1,\dots;m_1,\dots)}$  has a scaling dimension  $h_{N,s} + l$ , and the character is independent of the  $S_z$  quantum number  $\sigma$ . Mathematically, this is the character for the irreducible representation of the  $U(1)$  KM algebra and the Virasoro algebra with the primary field  $\Psi_{N,s,\sigma}$  as highest weight.) Since  $j$  acts only on  $\Psi_N = e^{-iN\sqrt{m}\phi}$  and  $T$  only on  $\psi_\sigma^s$ , the character of  $\Psi_{N,s,\sigma} = \Psi_N \psi_\sigma^s$  is the product of the characters of the  $\Psi_N$  and  $\psi_\sigma^s$ :

$$\text{ch}_{N,s}(\xi) = \text{ch}_N(\xi) \text{ch}_s(\xi) . \quad (42)$$

The character for the  $U(1)$  KM algebra is the same as (21)<sup>25</sup>:

$$\text{ch}_N(\xi) = \frac{\xi^{mN^2/2}}{\prod_n (1 - \xi^n)} . \quad (43)$$

The character,  $\text{ch}_s$ , in the  $c = -2$  model, is for the Virasoro representation with the highest weight  $[(4s + 1)^2 - 1]/8$  (conformal dimension of  $\psi_\sigma^s$ ). By applying formula (19) given in Ref. 25, we obtain

$$\text{ch}_s(\xi) = \frac{\xi^{h_s} - \xi^{h_{s+1/2}}}{\prod_n (1 - \xi^n)} . \quad (44)$$

We notice that this character is non-trivial, because of the existence of the null states.

Now let us summarize, in general terms but with the HR state as an example, our above procedure for generating edge excitations of a non-abelian FQH state using OPA techniques. For an FQH state that admits a OPA description<sup>18</sup>, in general, the electron operator(s) contain two factors, the abelian part and non-abelian part (see (28)). The abelian part is a vertex operator in a Gaussian model, and the non-abelian part, e.g. for the HR state, is  $\psi_\pm$  which generate a  $c = -2$  CFT. As discussed in Ref. 18,  $\psi_\pm$  in the non-abelian part generate a closed OPA, called the center algebra, through their OPE. The center algebra contains the identity operator,  $\psi_\pm$  and their descendants, and all primary fields generated by the OPE of  $\psi_\pm$  (which in the HR case are the spin- $s$  fields  $\psi_\sigma^s$ ) and their descendants. According to our chiral OPA description, the holomorphic part of the FQH wave function (of zero energy) can be written as a correlation between  $\psi_{e\pm}$ 's with an operator  $\Psi$  inserted at infinity. The insertion  $\Psi$  also contains two factors, the abelian part and a non-abelian part. The non-abelian part can be any operator in the center algebra and the abelian part can be any descendent of  $\Psi_N$  under the  $U(1)$  current  $j$ . If we choose the non-abelian part of  $\Psi$  to be the identity operator and the abelian part to be  $\Psi_N = e^{-iN\sqrt{m}\phi(z)}$ , then  $\Psi = \Psi_N$  will generate the ground state wave function of the FQH state. If the non-abelian part is chosen to be some other operators in the center algebra and/or the abelian part is chosen to be an descendent of  $\Psi_N$ , then the insertion  $\Psi$  will generate the edge excitations.<sup>27</sup>

Having edge states generated in this way, two questions immediately come to our mind: First, are the edge states generated with different descendants linearly independent to each other? Second, do the insertions with all descendant fields in the center algebra and the  $U(1)$  current algebra exhaust all possible edge states? These are very hard questions. Let us examine them in turns.

For the first question, obviously the descendants of primary fields with different dimensions (which are related to the angular momentum quantum numbers) or different spin quantum numbers generate linearly independent edge states. The hard part of the

question is whether different descendants of the same spin quantum number at the same level will generate linearly independent states or not. Generally in CFT, linearly independent descendants, as operators, should have different (or linearly independent) sets of correlations which contain *arbitrary* numbers of electron operators  $\psi_{\pm}$ . But when the number,  $N$ , of electrons is finite and fixed, one can *not* claim that different descendant fields generate linearly independent correlations, in particular for descendant fields at arbitrarily large levels. So we suggest that the insertions with different operators in the center algebra generate different edge excitations in the thermodynamic limit or in the large- $N$  limit; though at the moment we do not know how to prove it within CFT. With the help of the suggested correspondence between the descendant fields and the edge states (in the large  $N$  limit), one can use the known results about the descendant fields derived from the KM algebra and the center algebra to obtain the spectrum of low-lying edge states (see below). We have done numerical diagonalization for small systems to test the predictions. As will be seen in Sec. 4, indeed the numerical results shows, on one hand, the violation of the suggested correspondence for finite  $N$  at large level  $l$ . On the other hand, they verify the validity of the correspondence at the levels less than a certain number of order  $N$  in all FQH states we have considered.

Now we turn to the question of whether the descendant fields discussed above can generate all possible edge excitations in the system. We would like to first point out that the edge wave functions constructed above not only preserve the structure of zeros in the ground state as two electrons approach each other, they may also preserve the structure of higher-order zeros for three or more electrons approaching each other. If the wave functions generated by the descendant fields do not exhaust the zero-energy sector of a two-body Hamiltonian, in principle it is possible to construct a more restrictive Hamiltonian that contains, in addition to the two-body interaction shown in (26), also three-body, four-body, *etc.* interactions, so that the new Hamiltonian makes the wave functions constructed above be and exhaust its zero-energy edge-excited states. Thus, the question of whether the space of edge states contains more states depends on the dynamics of electron interactions, and cannot be addressed by merely studying the wave functions. In the following, we will assume that the Hamiltonian satisfies the above conditions.

Under the assumption that the descendant fields and the edge states have a one-to-one correspondence and the assumption that the descendant fields exhaust all edge excitations, the space of the edge excitations,  $V_{edge}$ , can be written as

$$V_{edge} = V_{U(1)} \otimes V_{ca} \quad (45)$$

where  $V_{U(1)}$  is the space of states of the  $U(1)$  KM algebra generated by  $j_n$ , and  $V_{ca}$  the space of states of the center algebra generated by  $\psi_{\pm}$ . Let us introduce the character for edge excitations

$$\text{Ch}(\xi) \equiv \sum_L D_L \xi^L, \quad (46)$$

where  $D_L$  is the number of the edge states with angular momentum  $L$ . From (40) and (41), we see that the edge excitations of an  $N$ -electron system with total spin  $s$  and a fixed  $S_z$  component  $\sigma$  are described by the following character

$$\text{Ch}_{N,s}(\xi) = \text{ch}_{N,s}(\xi) \xi^{-Nh_e} = \frac{1 - \xi^{2s+1}}{\prod_n (1 - \xi^n)^2} \xi^{M_0^{(s)}}, \quad (47)$$

where  $M_0^{(s)} = h_{N,s} - Nh_e$  is the minimum angular momentum in the spin- $s$  sector. In section 4 we will test the edge-state spectrum (47) for the Hamiltonian (26) by numerical diagonalization.

The same approach can also be applied to the p-wave<sup>13,17,28</sup> and d-wave<sup>18</sup> paired FQH state of spinless electrons. Here we only present the final results.

Both pairing states contain a  $Z_2$  structure, because of a similar structure in their chiral OPA (see Ref. 18), corresponding to an even or odd number of electrons. The center algebra of (the non-abelian part of) the p-wave paired state is generated by the dimension-1/2 (Majorana) fermion field in the Ising model, which contains an even(odd)-sector generated by an even (odd) number of fermion fields. (The fermion number is not conserved, but is *mod 2* conserved.) The character for the two sectors is given by that for the  $c = 1/2$  Virasoro representation with the identity ( $h = 0$ ) and the fermion field ( $h = 1/2$ ) as the highest weight respectively. Using the formulas (21-23) in Ref. 25, we obtain

$$\begin{aligned} \text{ch}_{\text{even}} &= \frac{\sum_{m \in \mathbb{Z}} \xi^{12m^2+m} (1 - \xi^{6m+1})}{\prod_{n>0} (1 - \xi^n)} \quad , \\ \text{ch}_{\text{odd}} &= \xi^{1/2} \frac{\sum_{m \in \mathbb{Z}} \xi^{12m^2+5m} (1 - \xi^{6m+2})}{\prod_{n>0} (1 - \xi^n)} \quad . \end{aligned} \quad (48)$$

Thus the characters for edge excitations of the p-wave paired state, after including the Gaussian (or abelian) part, are

$$\begin{aligned} \text{Ch}_e &= \frac{\sum_{m \in \mathbb{Z}} \xi^{12m^2+m} (1 - \xi^{6m+1})}{\prod_{n>0} (1 - \xi^n)^2} \xi^{M_0^{(e)}} \quad , \quad \text{for even number of electrons;} \\ \text{Ch}_o &= \frac{\sum_{m \in \mathbb{Z}} \xi^{12m^2+5m} (1 - \xi^{6m+2})}{\prod_{n>0} (1 - \xi^n)^2} \xi^{M_0^{(o)}} \quad , \quad \text{for odd number of electrons.} \end{aligned} \quad (49)$$

where  $M_0^{e,o}$  is the angular momentum for the ground state with an even or odd number of electrons.

As for the d-wave paired state, the center algebra (for the non-abelian part) is generated by a dimension-1  $U(1)$  current  $\tilde{j}$  (which is different from the  $U(1)$  current  $j$  for the abelian part). In Ref. 18 it was shown that the chiral OPA of this  $c = 1$  model is a  $Z_2$  orbifold model. Each of the two sectors generated by even or odd number of  $\tilde{j}$ 's may contain a lot of Virasoro representations, since the OPE of two  $\tilde{j}$ 's may generate higher integer-dimensional primary fields. In the Appendix, we show that the characters in the two sectors are given by

$$\text{ch}_{\text{even}} = \frac{\sum_{m \geq 0} (-)^m \xi^{m^2}}{\prod_{n>0} (1 - \xi^n)} \quad , \quad \text{ch}_{\text{odd}} = \frac{\sum_{m \geq 1} (-)^{m-1} \xi^{m^2}}{\prod_{n>0} (1 - \xi^n)} \quad . \quad (50)$$

Note the sum of the above two characters is just the character of the  $U(1)$  KM algebra  $1/\prod_{n>0} (1 - \xi^n)$ . The chiral OPA of the  $U(1)/Z_2$  model plus the abelian  $U(1)$  part leads to the following characters for edge excitations of the d-wave paired state:

$$\begin{aligned} \text{Ch}_e(\xi) &= \frac{\sum_{m \geq 0} (-)^m \xi^{m^2}}{\prod_{n>0} (1 - \xi^n)^2} \xi^{M_0^{(e)}} \quad , \quad \text{for even number of electrons;} \\ \text{Ch}_o(\xi) &= \frac{\sum_{m \geq 1} (-)^{m-1} \xi^{m^2-1}}{\prod_{n>0} (1 - \xi^n)^2} \xi^{M_0^{(o)}} \quad , \quad \text{for odd number of electrons.} \end{aligned} \quad (51)$$

Again in next section we will present a numerical test of these predictions by directly diagonalizing the p-wave and d-wave Hamiltonians given in Ref. 17 and Ref. 18. As we will see, the numerical results for the first a few low-lying edge states completely agree with the CFT results (47), (49) and (51). This suggests that the chiral OPA algebra is an effective and powerful tool to study edge excitations of non-abelian FQH states.

#### 4. NUMERICAL RESULTS AND SPECIFIC HEAT

In this section we would like to compare the above chiral OPA predictions with results of numerical diagonalization. The edge excitations for the p-wave paired FQH state have been studied in Ref. 17. The edge spectrum of states with low-lying angular momenta obtained numerically agrees exactly with (49) from chiral OPA.

For the HR state, according to (47), the number of edge excitations in the spin- $s$  sector at angular momentum  $L = M_0^s + l$ , with  $M_0^s = \frac{m}{2}N(N-1) + h_s - N$  (the minimum angular momentum in the spin- $s$  sector) and  $0 \leq l \leq 5$ , is given by

$$\begin{array}{rcccccc}
 l: & 0 & 1 & 2 & 3 & 4 & 5 & \text{spin} \\
 & 1 & 1 & 3 & 5 & 10 & 16 & s = 0 \\
 & 1 & 2 & 4 & 8 & 15 & 26 & s = 1/2 \\
 & 1 & 2 & 5 & 9 & 18 & 31 & s = 1 \\
 & 1 & 2 & 5 & 10 & 19 & 34 & s = 3/2
 \end{array} \tag{52}$$

We have directly diagonalized of the Hamiltonian (26) in the symmetric gauge. The energy spectrum is labeled by the total angular momentum. The energy eigenstates can be divided into two classes. The first class is the zero-energy states that starts at the angular momentum  $M_0^s$ . Those states are identified as the edge excitations. The second class is the bulk excitations with finite energies. There is a clear energy gap that separates the edge excitations and the bulk excitations, as an evidence for the incompressibility of the HR state. By counting the zero-energy states of  $N$  electron system, we find the following edge spectrum

$$\begin{array}{rcccccc}
 l: & 0 & 1 & 2 & 3 & 4 & \text{spin} & \text{number} \\
 & 1 & 1 & 3 & 5 & 9 & s = 0 & N = 6 \\
 & 1 & 2 & 4 & 8 & 14 & s = 1/2 & N = 7 \\
 & 1 & 2 & 5 & 8 & 14 & s = 1 & N = 6 \\
 & 1 & 2 & 5 & 9 & & s = 3/2 & N = 7
 \end{array} \tag{53}$$

The discrepancy between (52) and (53) for large  $l$  is an effect due to finite  $N$ , as discussed in the previous section. However, for  $l \leq N_m \equiv (N-2s)/2$  the edge spectrum in the spin- $s$  sector has reached the thermodynamical values (*i.e.*, those which do not change when we increase  $N$ ). From our numerical results for different values of  $N$ , we also find that the number of the edge states at  $l = N_m + 1$  is always just one less than its thermodynamical value. From (53), we see that the thermodynamical values of the edge spectrum, at least for low-lying angular momentum states, well agree with the chiral OPA prediction (52).

For the d-wave paired state for spinless electrons, according to (51), the number of edge excitations for an even (or odd) number of electrons with low-lying angular momenta  $L = M_0^{(e)} + l$  (or  $L = M_0^{(o)} + l$ ) is given by

$$\begin{array}{rcccccc}
 l: & 0 & 1 & 2 & 3 & 4 & 5 & \text{sector} \\
 & 1 & 1 & 3 & 5 & 11 & 18 & \text{even} \\
 & 1 & 2 & 5 & 9 & 18 & 31 & \text{odd}
 \end{array} \tag{54}$$

Direct diagonalizations of the d-wave Hamiltonian in Ref. 18 for  $N$  electrons give the following edge spectrum

$$\begin{array}{rcccccc}
l: & 0 & 1 & 2 & 3 & 4 & 5 & \text{number} \\
& 1 & 1 & 3 & 5 & 10 & 15 & N = 6 \\
& 1 & 2 & 5 & 9 & 17 & & N = 7
\end{array} \tag{55}$$

For  $l \leq N_m \equiv [N/2]$  the numbers of the edge states have reached their thermodynamical value. The number of the edge states at  $l = N_m + 1$  again is always just one less than its thermodynamical value. We see again that the thermodynamical values of the edge spectrum, at least for low-lying angular momenta, well agree with the results (51) from chiral OPA.

Here we would like to remark that the edge excitations in our calculations have zero energy, because the our Hamiltonians do not contain confining potential. If we include a parabolic confining potential, then the Hamiltonian will receive a term  $V_c \hat{L}$  where  $\hat{L}$  is the total angular momentum operator. Thus the parabolic confining potential only shifts the energy without changing the wave function. The edge excitations will have a non-zero energy  $V_c L = V_c(M_0 + l)$ . But the degeneracy at each total angular momentum remains exactly the same. For more general confining potential the degeneracy at a fixed angular momentum may be lifted. Even in this case the edge excitations and the bulk excitations remain well-separated. This is because the energy  $E_l$  of the edge excitations of level  $l$  is proportional to the inverse of the length of the edge:  $E_l \propto l/\sqrt{N}$ . When  $l \ll \sqrt{N}$  the energy of the edge excitations is much less than the bulk energy gap, and the edge and the bulk excitations are still well-separated. In this case the chiral OPA, instead of generating zero-energy states, generates low-lying edge excitations.

Finally let us calculate the specific heat of the edge excitations from the character formula. Assume all edge excitations have the same velocity  $v$ . The energy of edge excitations in the angular momentum  $L$  sector is given by

$$E = v(L - M_0)/R, \tag{56}$$

where  $M_0$  is the angular momentum of the ground state and  $R$  the radius of the FQH droplet. Thus the partition function of edge excitations is directly related to the character of the edge excitations (see (46)):

$$Z = \sum_{l=0} D_{l+M_0} e^{-l\beta\frac{v}{R}} = \text{Ch}(\xi)\xi^{-M_0}, \tag{57}$$

with  $\xi = e^{-\beta\frac{v}{R}}$ . For one-dimensional systems, the number of states,  $N_n \equiv D_{n+M_0}$ , at level  $n$ , has the following asymptotic form

$$N_n \sim A n^\eta \exp\left(\sqrt{\frac{2\pi^2}{3}cn}\right), \tag{58}$$

where  $c$ ,  $A$ , and  $\eta$  are constants. From (58) it follows that the specific heat per unit length is

$$C = c \frac{\pi T}{6 v}, \tag{59}$$

independent of exponent  $\eta$  of the prefactor in (58). The constant  $c$  is known to be unity for the Gaussian model<sup>29</sup>. We will call  $c$  the number of the edge branches, since  $c = n$  if edge excitations are described by  $n$  branches of chiral phonons (*i.e.*, by  $U(1)^n$  KM algebra).

The edge excitations for the d-wave FQH state contain a  $U(1)$  part and a non-abelian part. Let us concentrate on the non-abelian part. First the specific heat of the non-abelian sector cannot be larger than that of the Gaussian model:  $C \leq \frac{\pi T}{6v}$ , since the number of the edge excitations at each angular momentum is less than that of the Gaussian model. We also see from the character formula (50) that the number of the d-wave edge excitations at level  $n$  satisfies  $N_n^d > N_n^G - N_{n-1}^G$ , where  $N_n^G \sim A_G n^{\eta_G} \exp(\sqrt{\frac{2\pi^2}{3}}n)$  is the number of the edge excitations in the Gaussian model ( $N_n^G$  is the number of partitions of the integer  $n$ ). Thus  $N_n^d > A' n^{\eta_G - \frac{1}{2}} \exp(\sqrt{\frac{2\pi^2}{3}}n)$ . This implies  $C \geq \frac{\pi T}{6v}$ . So we have  $C = \frac{\pi T}{6v}$ . Including the  $U(1)$  part, the d-wave FQH state has a total of two branches of edge excitations.

For the HR state, if we fix  $S_z = 0$ , the discussions in the last paragraph also apply. We find the specific heat for non-abelian part of the edge excitations to be  $C = \frac{\pi T}{6v}$ . Hence the HR state also has two branches of edge excitations. We note that in our present case, the number of branches of edge excitations,  $c = 2$ , is not the same as the central charge  $(-2 + 1)$  of the bulk CFT.

## 5. DISCUSSIONS

In this paper we have proposed that there is a close connection between the spectrum of FQH edge excitations and the descendent fields generated by  $U(1)$  current algebra and the center algebra (the chiral OPA generated by the non-abelian part of the electron operator). This connection allows us to apply CFT techniques to construct wave functions for edge excitations, and use algebraic methods to enumerate the edge states. The predictions of the edge spectra from the chiral OPA were confirmed by numerical calculations for all FQH states studied.

Previously a similar connection between the (minimal) chiral OPA and the bulk FQH wave function was studied in detail in Ref. 18. Together with the connection proposed in this paper between the (minimal) chiral OPA and the FQH edge excitations, we see that indeed chiral OPA provides us a unified description of bulk wave function, quasiparticle excitations, and edge excitations. This indicates that the topological orders in the FQH systems are characterized by chiral OPA.

To conclude, a few remarks are in order. First, the  $c = -2$  CFT in our discussion of the HR state is known to contain negative-norm states, while all bulk and edge excitations in the HR state, of course, have positive norm. This indicates clearly that the inner product between physical edge states and that between the corresponding states in the CFT are not the same. The chiral OPA in the CFT is used only to generate the wave functions and spectrum of edge excitations; nothing is implied by this for the inner product of the physical edge states. We also like to point out that our construction of the edge states relies only on the chiral OPA. One can, in principle, construct edge states and calculate the edge characters directly from the chiral OPA, as we did in the appendix, without even mentioning the Virasoro algebra. For the FQH states we studied in this paper, the chiral OPA happen to form a representation of the Virasoro algebra. This additional information allows us to use the well-known character formula for the Virasoro algebra to calculate the edge characters.

Near the completion of the present paper, we learned that one can also use a scalar

fermion theory to construct edge excitations of the HR state.<sup>30</sup> We would like to thank N. Read for communicating his result prior to publication.

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## APPENDIX

Let us first consider the character of the descendent fields of the identity generated by the  $U(1)$  current  $j$  (see (21)). Note the character for states generated by  $j_m$  and its power (with  $m$  fixed) is given by  $\sum_{n=0}^{\infty} (\xi^m)^n$ . Thus the total character has a form

$$\prod_{m=1}^{\infty} \sum_{n=0}^{\infty} (\xi^m)^n = \frac{1}{\prod_{m>0} (1 - \xi^m)}. \quad (60)$$

The character counts the number of states generated by  $j$  at each level. In the phonon language, the level is proportional to the total energy of phonons created by  $j_{-m}$ 's. Thus the character counts the degeneracy at each energy level.

To obtain the character of the descendent fields (or the character of states) generated by an even or odd numbers of  $j$ , we may consider

$$f(\xi, \eta) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (\eta \xi^m)^n = \frac{1}{\prod_{m>0} (1 - \eta \xi^m)} \quad (61)$$

Here we use the power of  $\eta$  to count the number of the  $j$ -operators used to create a particular descendent field (or a state). Thus the characters for the even and odd sectors are given by

$$\text{ch}_{\text{even}} = \frac{1}{2} [f(\xi, 1) + f(\xi, -1)], \quad \text{ch}_{\text{odd}} = \frac{1}{2} [f(\xi, 1) - f(\xi, -1)]. \quad (62)$$

Using an identity

$$\frac{1}{2} \prod_{n>0} \frac{1 - \xi^n}{1 + \xi^n} + \frac{1}{2} = \sum_{n=0}^{\infty} (-)^n \xi^{n^2} \quad (63)$$

we obtain (50) from (62). The identity (63) can be obtained from a  $\theta$ -function identity  $\theta_4(0; 2\tau) = [\eta(\tau)]^2 / \eta(2\tau)$ , where  $\eta(\tau)$  is the Dedekind function. (See Ref. 31)

We know the OPE of  $j$  generate many primary fields of the Virasoro algebra. In fact those primary fields  $\phi_n$  can be labeled by an integer  $n = 0, 1, \dots$ . The dimension of  $\phi_n$  is  $h_n = n^2$ .  $\phi_0$  is the identity and  $\phi_1$  is the current operator. Comparing the character formula of  $\phi_n$  (see the theorem 5 in Sec. 3 of Ref. 25) with (50), we find that  $\phi_n$  carries a  $Z_2$ -charge  $(-)^n$ .

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