

# Theory of the edge states in fractional quantum Hall effects\*

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ABSTRACT: The dynamical theory of the edge excitations of generic fractional quantum Hall (FQH) states is summarized and expanded. The low energy effective theory of the edge excitations for the most general abelian FQH states (including spin-unpolarized and multi-layer FQH states) and some non-abelian FQH states is derived using several different methods. The propagators of the electrons and the quasiparticles are calculated for the above FQH states. The microscopic theory of the edge excitations for the Laughlin states is also presented. Some simple applications of the edge theory to the transport properties of the FQH states are discussed. In particular, the tunneling between edge states is shown to be a powerful tool to probe the internal topological orders in the FQH states. It can be used to distinguish different FQH states with the same filling fraction and to detect the non-abelian FQH states in experiments.

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## I. INTRODUCTION

Fractional quantum Hall effects (FQHE) discovered by Tsui, Stormer and Gossard<sup>1</sup> open a new era in theory of strongly correlated system. For the first time we have to completely abandon the theories based on the single-body picture (such as Fermi liquid theory) and use an intrinsic many-body theory proposed by Laughlin<sup>2</sup> and others<sup>3</sup> to describe FQHE. Due to the repulsive interaction and strong correlation between the electrons, a FQH liquid is an incompressible state despite the first Landau level is only partially filled. All the bulk excitations in the FQH states have finite energy gaps. The FQH states and insulators are very similar in the sense that both states have finite energy gap and short ranged electron propagators. Because of this similarity, people were puzzled by the fact that the FQH systems apparently have very different transport properties than ordinary insulators. Halperin first pointed out that the integral quantum Hall (IQH) states contain gapless edge excitations.<sup>4</sup> Although the electronic states in the bulk are localized, the electronic states at the edge of the sample are extended (*i.e.*, the electron propagator along the edge is long ranged).<sup>5</sup> Therefore the nontrivial transport properties of the IQH states come from the gapless edge excitations.<sup>4,6</sup> *e.g.*, a two probe measurement of a QH sample can result in a finite resistance only when the source and the drain are connected by the edges. If the source and the drain are not connected by any edge, the two probe measurement will yield an infinite resistance at zero temperature, a result very similar to the insulators. The edge transport picture has been supported by many experiments.<sup>7</sup> Halperin also studied the dynamical properties of the edge excitations of the IQH states and found the edge excitations are described by a chiral 1D Fermi liquid theory.

Using the gauge argument in Ref. 8,4,9, one can easily show that FQH states also support gapless edge excitations. Thus it is natural to conjecture that the transport in FQH states is also governed by the edge excitations.<sup>10,11</sup> However since FQH states are intrinsically many-body states, the edge excitations in the FQH states cannot be constructed from a single-body theory. Or in another word the edge excitations of FQH states should not be described by Fermi liquid. In this case we need completely new approaches to understand the dynamical properties of the edge states of FQHE. Recent advances in fabrication of small devices make it possible to study in detail the dynamical properties of the edge states in FQHE. Thus it is very important to develop a quantitative theory for edge states in FQHE to explain new experimental data.

There is another motivation to study the edge states in FQHE. We know that different FQH states were generally labeled by their filling fractions. However now it becomes clear that FQH states contain extremely rich internal structures that the filling fraction alone is not enough to classify all the different universality classes of FQH states.<sup>12,13</sup> One can easily construct different FQH states with the same filling fraction.<sup>14,15,16,17</sup> Now we are facing two problems: A) How to label (or classify) different universality classes of FQH states? B) How to measure the internal structures of FQH states experimentally. Or in another word how to distinguish two different FQH states with the same filling fraction?

One way to resolve A) is to use the various construction schemes to label different FQH states. However this approach is no good because the construction schemes and the universality classes of FQH states do not have a one to one correspondence. Different construction schemes some times lead to the same FQH state. One may also try to use symmetries to classify different FQH states as we did before for many other condensed matter states. This does not work either because it was shown that different FQH states are not distinguished by their symmetries.<sup>13</sup> In the fact the FQH states represent a new

kind of universality classes that we have not encountered. A concept of topological order is introduced to describe this new kind of ordering.<sup>12,13</sup> Recent study<sup>18,19,20</sup> show that FQH states can be divided into two classes: abelian FQH states in which quasiparticles have only Abelian statistics, and non-abelian FQH states in which some quasiparticles have non-Abelian statistics. It was shown that the abelian FQH states can be labeled by integer valued symmetric matrices.<sup>15,16,17,21</sup> For example, the hierarchy states constructed by Haldane and Halperin are classified by tri-diagonal matrices with off diagonal elements  $\pm 1$ . One can even show that the topological orders (or the universality classes) in the most general abelian FQH states are classified<sup>21</sup> by the integer valued symmetric matrices with odd diagonal elements.

Some physical measurements of the topological orders in FQH states through the ground state degeneracy and the non-Abelian Berry's phases of FQH states on closed Riemann surfaces were discussed in Ref. 12,13. However experimentally one can never put an HQ state on a closed Riemann surface (say a sphere). Thus results in Ref. 12,13 can only be checked in numerical calculations. The study in Ref. 22,23 indicates that edge excitations in FQH states provide an important (probably the only practical) probe to detect the topological orders in the bulk FQH states. Thus the question B) can be answered through the edge states. Using the edge excitations we also can tell whether a FQH state is an abelian FQH state or a non-abelian FQH state. The edge states provide us a practical window through which we can look into the internal structures in FQH states. The measurements of the edge states can provide us new quantum numbers, in addition to the filling fractions, to characterize different quantum Hall states.

In this paper we are going to summarize and expand the results obtained in Ref. 9,24,22,25,19. We will derive the low energy effective theory of the edge excitations for most general abelian FQH states and some non-abelian FQH states. We will also study the generic quasiparticle and the electron operators and their propagators on the edges of the above FQH states. Some simple applications of the edge theory to the transport properties of the FQH states will also be discussed.

For readers who just like to get a simple picture about the edge excitations in the FQH states may choose to read sections 2.1, 2.3 and chapter 6. Chapter 4 contains a microscopic theory of the edge excitations in the Laughlin states. The generic structures of the edge excitations of the (generalized) hierarchical states are discussed in section 2.5 and chapter 3. The edge theory of some non-abelian FQH states is constructed in chapter 5 that extensively uses the parton construction discussed in section 2.4.

## 2. CHIRAL DYNAMICS OF THE EDGE EXCITATIONS AND THE KAC-MOODY ALGEBRA

In this section we will review various approach to the dynamics of the edge excitations. Effects of interactions between edge states will be discussed as well.

### 2.1 Hydrodynamical approach to the edge excitations

The simplest (but not complete) way to understand the dynamics of the edge excitations is to use the hydrodynamical approach. In this approach, one use the fact that

QH (IQH or FQH) states are incompressible irrotational liquid that contain no low energy bulk excitations. Therefore the only low lying excitations (below the bulk energy gap) are surface waves on a HQ droplet. These surface waves are identified as edge excitations of the HQ state.<sup>26,27,25</sup>

In the hydrodynamical approach we first study the classical theory of the surface wave on the HQ droplet. Then we quantize the classical theory to obtain the quantum description of the edge excitations. It is amazing that the simple quantum description obtained from the classical theory provides a complete description of the edge excitations at low energies that allow us to calculate the electron and the quasiparticle propagators along the edges.

Consider a QH droplet with filling fraction  $\nu$  confined by a *smooth* potential well (see Fig. 1). Due to the non-zero conductance, the electric field of the potential well generate a persistent current flowing along the edge:

$$\vec{j} = \sigma_{xy} \hat{z} \times \vec{E}, \quad \sigma_{xy} = \nu \frac{e^2}{h} \quad (2.1)$$

This implies that the electrons near the edge drift with a velocity

$$v = \frac{E}{B} c \quad (2.2)$$

where  $c$  is the velocity of the light. Thus the edge wave also propagates with the velocity  $v$ . Let us use one dimensional density  $\rho(x) = nh(x)$  to describe the edge wave, where  $h(x)$  is the displacement of the edge,  $x$  is the coordinate along the edge, and  $n = \frac{\nu}{2\pi l_B^2}$  is the two dimensional electron density in the bulk. We see that the propagation of the edge waves are described by the following wave equation:

$$\partial_t \rho - v \partial_x \rho = 0 \quad (2.3)$$

Notice that the edge waves always propagate in one direction, there are no waves that propagate in the opposite direction.

The Hamiltonian (*i.e.*, the energy) of the edge waves is given by

$$H = \int dx \frac{1}{2} e^2 h \rho E = \int dx \pi \nu v \rho^2 \quad (2.4)$$

In the momentum space (2.3) and (2.4) can be rewritten as

$$\begin{aligned} \dot{\rho}_k &= i v k \rho_k \\ H &= 2\pi \frac{v}{\nu} \sum_{k>0} \rho_k \rho_{-k} \end{aligned} \quad (2.5)$$

where  $\rho_k = \int dx \frac{1}{\sqrt{L}} e^{ikx} \rho(x)$ , and  $L$  is the length of the edge. Compare (2.5) with the standard Hamiltonian equation

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad (2.6)$$

we find that if we identify  $\rho_k|_{k>0}$  as the “coordinates”, then the corresponding canonical “momenta” can be identified as  $p_k = i2\pi\rho_{-k}/\nu k$ . We would like to stress that because the edge waves are chiral, the displacement  $h(x)$  contains both the “coordinates” and the “momenta”.

Knowing the canonical coordinates and momenta, it is easy to quantize the classical theory. We simply view  $\rho_k$  and  $p_k$  as operators that satisfy  $[p_k, \rho_{k'}] = i\delta_{kk'}$ . Thus after quantization we have

$$\begin{aligned} [\rho_k, \rho_{k'}] &= \frac{\nu}{2\pi} k \delta_{k+k'} \\ k, k' &= \text{integer} \times \frac{2\pi}{L} \\ [H, \rho_k] &= \nu \rho_k \end{aligned} \tag{2.7}$$

The above algebra is called the ( $U(1)$ ) Kac-Moody (K-M) algebra.<sup>28</sup> A similar algebra has also appeared in the Tomonaga model.<sup>29</sup> Notice that (2.7) just describes a collection of decoupled harmonic oscillators (generated by  $(\rho_k, \rho_{-k})$ ). Thus (2.7) is an one dimensional free phonon theory (with only a single branch of phonon) and is exactly soluble. We will show later that (2.7) provides a complete description of the low lying edge excitations of the HQ state.

To summarize, we find that the edge excitations in the QH states are described by a free (chiral) phonon theory at low energies. We not only show the existence of the gapless edge excitations, we also obtain the density of states of the edge excitations. The specific heat (per unit length) of the edge excitations is found to be  $\frac{\pi}{6} \frac{T}{v}$ . The edge excitations considered here do not change the total charge of the system and hence are neutral. In the following, we will discuss the charged excitations and calculate the electron propagator from the K-M algebra (2.7).

The low lying charge excitations obviously correspond to adding (removing) electrons to (from) the edge. Those charged excitations carry integer charges and are created by electron operators  $\Psi^\dagger$ . The above theory of the edge excitations is formulated in terms of 1D density operator  $\rho(x)$ . So the central question is to write the electron operator in terms of the density operator. The electron operator on the edge create a localized charge and should satisfy

$$[\rho(x), \Psi^\dagger(x')] = \delta(x - x') \Psi^\dagger(x') \tag{2.8}$$

Since  $\rho$  satisfy the Kac-Moody algebra (2.7), one can show that the operators that satisfy (2.8) are given by<sup>24</sup>

$$\Psi \propto e^{i\frac{1}{\nu}\phi} \tag{2.9}$$

where  $\phi$  is given by  $\rho = \frac{1}{2\pi} \partial_x \phi$ .

(2.8) only implies that the operator  $\Psi$  carry charge  $e$ . In order to identify  $\Psi$  as an electron operator we need to show that  $\Psi$  is a fermionic operator. Using the K-M algebra (2.7) we find that

$$\Psi(x)\Psi(x') = (-)^{1/\nu} \Psi(x')\Psi(x) \tag{2.10}$$

We see that the electron operator  $\Psi$  in (2.9) is fermionic only when  $1/\nu = m$  is an odd integer in which case the QH state is a Laughlin state.<sup>24,30</sup>

In the above discussion we have made an assumption that is not generally true. We have assumed that the incompressible QH liquid contain only *one* component of incompressible fluid which leads to one branch of edge excitations. The above result implies that,

when  $\nu \neq 1/m$ , the edge theory with only one branch do not contain the electron operators and is not self consistent. Therefore we conclude that the FQH states with  $\nu \neq 1/m$  must contain more than one branch of edge excitations. (Here we have ignored the possibility of the pairing between the electrons.<sup>31</sup>) Later we will see that the one-branch assumption is true only for the simple Laughlin states with filling fraction  $\nu = 1/m$ . In hierarchical FQH states, there are several condensates that correspond to several components of incompressible fluid. Each component gives rise to a branch of the edge excitations. Thus a generic QH state may contain many branches of the edge excitations,<sup>11,22</sup> even when electrons are all in the first Landau level.

Now let us calculate electron propagator along the edge of the Laughlin states with  $\nu = 1/m$ . In this case the above simple edge theory is valid. Because  $\phi$  is a free phonon field with a propagator

$$\langle \phi(x, t)\phi(0) \rangle = -\nu \ln(x - vt) + \text{const.} \quad (2.11)$$

the electron propagator can be easily calculated as<sup>24</sup>

$$G(x, t) = \langle T(\Psi^\dagger(x, t)\Psi(0)) \rangle = \exp\left[\frac{1}{\nu^2} \langle \phi(x, t)\phi(0) \rangle\right] \propto \frac{1}{(x - vt)^m} \quad (2.12)$$

Another way to calculate the electron propagator can be found in Ref. 32, where the oscillator formalism is used.

The first thing we see is that the electron propagator on the edge of FQH state acquires a non-trivial exponent  $m = 1/\nu$  that is not equal to one. This implies that the electrons on the edge of the FQH state are strongly correlated and cannot be described by Fermi liquid theory. We will call such a new type of electron state chiral Luttinger liquid.

The K-M algebra (2.7) and the electron operator (2.9) provided a complete description of both neutral and charged edge excitations at low energies. We would like to remark that the propagator (2.12) is correct only for large  $x$  and  $t$ . At short distance the form of the propagator depends on the details of the electron interactions and the edge potentials. We also like to emphasize that the exponent  $m$  of the edge propagator is determined by the bulk state. Such an exponent is a topological number that is independent of electron interactions, edge potential *etc.*. The quantization of the exponent is directly related to the fact that the exponent is linked to the statistics of the electrons (see (2.10)). Thus the exponent can be regarded as a quantum number that characterizes the topological orders in the bulk FQH states.

In the momentum space the electron propagator has a form

$$G(k, \omega) \propto \frac{(vk + \omega)^{m-1}}{vk - \omega} \quad (2.13)$$

The anomalous exponent  $m$  can be measured in tunneling experiments. The tunneling density of states of electron is given by

$$N(\omega) \propto |\omega|^{m-1} \quad (2.14)$$

This implies that differential conductance has a form  $\frac{dI}{dV} \propto V^{m-1}$  for a metal-insulator-FQH junction.

## 2.2 Gauge invariance and edge states

In this section we are going to show that the existence of the gapless edge excitations in the FQH states is a direct consequence of the gauge symmetry of the electromagnetic field. The dynamics of the edge states also can be derived from the combination of the gauge invariance and the locality of the FQH theory.<sup>9</sup>

Consider a QH state in a background magnetic field  $\bar{A}_i$  ( $\bar{A}_0 = 0$ ) with a Hall conductance  $\sigma_{xy} = \nu \frac{e^2}{h}$ . After integrating out the electrons, we obtain an effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{\nu e^2}{4\pi} \delta A_\mu \partial_\nu \delta A_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{4g_1^2} (\delta F_{0i})^2 - \frac{1}{4g_2^2} (\delta F_{12})^2 + \dots \quad (2.15)$$

where  $\delta A_\mu = A_\mu - \bar{A}_\mu$  is fluctuation around the constant magnetic field and  $\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$  is the field strength. The coefficient of the Chern-Simons term  $\delta A_\mu \partial_\nu \delta A_\lambda \epsilon^{\mu\nu\lambda}$  is given by the quantized Hall conductance.

On a compactified space, the action  $S_{\text{bulk}} = \int d^3x \mathcal{L}_{\text{eff}}(\delta A_\mu)$  is invariant under the gauge transformation. However, on a space with boundary, say, a disc  $D$ ,  $S_{\text{bulk}}$  is *not* gauge invariant:

$$S_{\text{bulk}}(\delta A_\mu + \partial_\mu f(x)) - S_{\text{bulk}}(\delta A_\mu) = \int dx_0 dx \frac{\nu e^2}{4\pi} f \delta F_{x0} \quad (2.16)$$

where  $x$  parametrizes the boundary of the disc. Because the microscopic theory is gauge invariant, (2.16) implies that  $S_{\text{bulk}}$  is not the complete action of the FQH states on the disc. Since the change in  $S_{\text{bulk}}$  is just a boundary term, the total gauge invariant effective action may be obtained by including a boundary action associated with the edge excitations  $S_{\text{tot}} = S_{\text{bulk}} + S_{\text{bd}}$ . Under the gauge transformation  $S_{\text{bd}}$  should transform as

$$S_{\text{bd}}(\delta A_\mu + \partial_\mu f) - S_{\text{bd}}(\delta A_\mu) = - \int dx_0 dx \frac{\nu e^2}{4\pi} f \delta F_{x0} \quad (2.17)$$

so that  $S_{\text{tot}}$  is gauge invariant. (2.17) implies that the current-current correlation function of the edge excitations  $K^{\alpha\beta}(t, x) = i \langle 0 | T(j^\alpha(t, x) j^\beta(0, 0)) | 0 \rangle$ , must satisfy (in  $k$ -space)

$$\begin{aligned} -k_\alpha K^{\alpha\beta} &= \frac{\nu e^2}{4\pi} \epsilon^{\alpha\beta} k_\alpha \\ K^{\alpha\beta}(k_\alpha) &= K^{\beta\alpha}(-k_\alpha) = K^{\alpha\beta*}(-k_\alpha) \end{aligned} \quad (2.18)$$

where  $\alpha, \beta = 0, x$ ,  $k_0 = \omega$ ,  $k_x = k = \frac{2\pi n}{L}$  is the momentum in  $x$  direction, and  $L$  is the length of the edge. We see that the complete action of the FQH states on a disc is given by  $S_{\text{bulk}}$  describing the bulk excitations *plus*  $S_{\text{bd}}$  describing the edge excitations.  $S_{\text{bulk}}$  and  $S_{\text{bd}}$  separately are not gauge invariant. But the total action  $S_{\text{tot}}$  is gauge invariant.

What we would like to do in the following is to use the edge current correlation  $K^{\alpha\beta}$  plus locality condition of the theory to determine some general dynamical properties of the edge states.

First we would like to show that the edge excitations must be gapless. If all the edge excitations had finite energy gap, then  $K^{\alpha,\beta}(\omega, k)$  would be a smooth function of  $\omega$  near  $\omega = 0$ . If we further assume that the theory is local,  $K^{\alpha,\beta}(\omega, k)$  would be a smooth function of  $k$  near  $k = 0$  (e.g.,  $K^{\alpha,\beta}(\omega, k)$  could not behave like  $\frac{F^{\alpha,\beta}(\omega)}{k}$  near  $k = 0$ ). However one can easily check that a smooth function of  $\omega$  and  $k$  (near  $\omega = 0$  and  $k = 0$ ) can never satisfy (2.18). Therefore, for local theories, the condition (2.18) implies the existence of gapless edge excitations.

Let us assume the edge excitations have many branches labeled by  $I$  and  $K^{\alpha,\beta}$  have poles at  $\omega = c_I k$  where  $c_I$  is the velocity of the  $I$ th branch. In this case the current correlation function  $K^{\alpha,\beta}$  takes the form: (for small  $(\omega, k)$ )

$$K^{\alpha,\beta}(\omega, k) = \sum_I \left[ \frac{\gamma_{I,k}^{\alpha\beta}}{\omega - c_I k + i\delta} - \frac{\gamma_{I,-k}^{\beta\alpha}}{\omega - c_I k - i\delta} \right] + P(\omega, k) \quad (2.19)$$

where  $P(\omega, k)$  is a polynomial of  $\omega$  and  $k$  that comes from the excitations with finite gap. Here  $\gamma_{I,k}^{\alpha\beta}$  is non-zero only when  $c_I k \geq 0$  since the excited states always have positive energies. Plugging (2.19) into (2.5) and using the fact that  $\gamma_{I,k}^{\alpha\beta}$  only depend on  $k$ , we find that  $K^{\alpha,\beta}$  must have a form (up to a polynomial in  $\omega$  and  $k$ )

$$K^{\alpha,\beta} = \begin{cases} \sum_I \frac{k \text{sgn}(c_I) \eta_I}{\omega - c_I k}, & (\alpha, \beta) = (0, 0) \\ \frac{1}{2} \sum_I \frac{\omega + c_I k}{\omega - c_I k} \text{sgn}(c_I) \eta_I, & (\alpha, \beta) = (x, 0), (0, x) \\ \sum_I \frac{c_I \omega \text{sgn}(c_I) \eta_I}{\omega - c_I k}, & (\alpha, \beta) = (x, x) \end{cases} \quad (2.20)$$

for small  $\omega$  and  $k$ , where

$$\sum_I \text{sgn}(c_I) \eta_I = \frac{\nu e^2}{2\pi}. \quad (2.21)$$

From (2.19), (2.20) and the fact that  $\gamma_{I,k}^{00} \geq 0$ , we find that  $\eta_I > 0$ .

Each term in the summations in (2.20) arises from gapless edge excitations with velocity  $c_I$ . It is convenient to write the current  $j^\alpha$  as a summation of  $j_I^\alpha$ :

$$j^\alpha = \sum_I j_I^\alpha \quad (2.22)$$

such that

$$K_{IJ}^{\alpha,\beta} \equiv \langle j_I^\alpha j_J^\beta \rangle = \delta_{IJ} \begin{cases} \frac{k \text{sgn}(c_I) \eta_I}{\omega - c_I k}, & (\alpha, \beta) = (0, 0) \\ \frac{1}{2} \frac{\omega + c_I k}{\omega - c_I k} \text{sgn}(c_I) \eta_I, & (\alpha, \beta) = (x, 0), (0, x) \\ \frac{c_I \omega \text{sgn}(c_I) \eta_I}{\omega - c_I k}, & (\alpha, \beta) = (x, x) \end{cases} \quad (2.23)$$

Therefore  $j_I^\alpha$  is associated with the gapless excitations with velocity  $c_I$ .  $j_{I,k}^\alpha$  generates a state with an energy  $\omega_k = c_I k$ :

$$H j_{I,k}^\alpha |0\rangle = c_I k j_{I,k}^\alpha |0\rangle \quad (2.24)$$

where  $j_{I,k}^\alpha = \int dx \frac{1}{\sqrt{L}} e^{ixk} j_I^\alpha(x)$ . From (2.23) we can obtain the vacuum expectation values of the commutator. We find that

$$\begin{aligned} \langle 0 | [j_{I,k'}^+, j_{J,k}^+] | 0 \rangle &= \text{sgn}(c_I) \eta_I k \delta_{k+k'} \delta_{IJ} \\ \langle 0 | [j_{I,k'}^-, j_{J,k}^+] | 0 \rangle &= \langle 0 | [j_{I,k'}^-, j_{J,k}^-] | 0 \rangle = 0. \end{aligned} \quad (2.25)$$

where  $j_I^\pm = \frac{1}{2}(j^0 \pm \frac{1}{c_I} j^x)$ .

Using the locality of the theory, we can further show that (see Ref. 9) (2.24) and (2.25) imply the operator equations

$$[H, j_{I,k}^\alpha] = c_I k j_{I,k}^\alpha \quad (2.26)$$

and

$$\begin{aligned} [j_{I,k'}^+, j_{J,k}^+] &= \text{sgn}(c_I) \eta_I k \delta_{k+k'} \delta_{IJ} \\ [j_{I,k'}^+, j_{J,k}^-] &= [j_{I,k'}^-, j_{J,k}^-] = 0 \end{aligned} \quad (2.27)$$

in the subspace of states with small momentum  $k$  (*i.e.*,  $kl_B \ll 1$ , where  $l_B$  is the magnetic length) and in the limit  $L \rightarrow \infty$ . Therefore the low lying edge excitations form a representation of several independent chiral  $U(1)$  Kac-Moody algebras.<sup>28</sup> The current algebra (2.27) determines the Hilbert space and (2.26) determines the dynamics of the low lying edge excitations. When there is only one branch, (2.26) and (2.27) are identical with (2.7) (notice that  $j^+$  is equal to  $\rho_k$  due to the charge conservation).

From the above discussion we find that the gauge invariance condition not only requires the existence of gapless edge excitations, it also determines some general dynamical properties of the edge states if the theory is local. In particular the specific heat of the edge excitations is proportional to  $T$  as implied by the K-M algebra. Due to its relation to the gauge invariance, the appearance of the gapless edge excitations is a general property of the QH states. The gapless edge excitations exist for arbitrary electron interactions, arbitrary edge potentials and any other perturbations. Certainly, using gauge invariance alone we cannot determine, say, the number of the edge branches of a FQH state. Such a property depends on the internal structures of the bulk state. To address this problem we need first to specify the topological orders in the FQH state.

We also like to point out that the above gauge argument only implies that the edge excitations must contain one or several  $U(1)$  K-M algebras to maintain the gauge invariance. This, however, does not imply that the  $U(1)$  K-M algebras describe all the gapless edge excitations. There are FQH states whose edge excitations contain, in addition to the K-M algebras, a neutral sector that decouples from the electromagnetic field.

### 2.3 The edge excitations of generic FQH states - hierarchical construction

In this section we are going to study structures of the edge states in generic FQH states. Traditionally, different FQH states were labeled by their filling fractions. However, it was pointed out that the filling fraction is not sufficient to specify the internal structures (*i.e.*, the topological orders) in the FQH states.<sup>13</sup> At a given filling fraction one can easily

construct FQH states with completely different topological orders.<sup>14,15,16,17,21</sup> Since the edge state structures are expected to depend on the topological orders in the bulk state, we need to specify the topological orders in the bulk state before we start to study the edge states. In the following we will specify the topological orders by explicit construction the FQH wave function. In this section we will consider the hierarchical construction. In the next section we will concentrate on the parton construction. A general derivation of the edge states from the topological orders will be discussed in the section 2.5.

Let us consider the  $\nu = \frac{2}{5}$  FQH state as an example. According to the hierarchical theory, the  $\nu = \frac{2}{5}$  FQH state is generated by the condensation of the quasiparticles in the  $\nu = \frac{1}{3}$  FQH state. To be definite let us consider a special edge potential as illustrated in Fig. 2. In this case the FQH state consists of two droplets, one is the electron condensate with filling fraction  $\frac{1}{3}$  and radius  $r_1$  and the other is the quasiparticle condensate with filling fraction  $\frac{1}{15}$  (note  $\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$ ) and radius  $r_2$ .

When  $r_1 - r_2 \gg l_B$ , the two edges are independent. Generalizing the hydrodynamical approach in section 2.1, we can show that there are two branches of the edge excitations whose low energy dynamics is described by

$$\begin{aligned} [\rho_{Ik}, \rho_{Jk'}] &= \frac{\nu_I}{2\pi} k \delta_{IJ} \delta_{k+k'} \\ H &= 2\pi \sum_{I,k>0} \frac{v_I}{\nu_I} \rho_{Ik} \rho_{I-k} \end{aligned} \quad (2.28)$$

where  $I = 1, 2$  labels the two branches,  $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{1}{15})$  are filling fractions of the electron condensate and the quasiparticle condensate, and  $v_I$  are the velocity of the edge excitations.  $\rho_I$  in (2.28) are the 1D electron densities given by  $\rho_I = h_I \nu_I \frac{2\pi}{l_B^2}$  where  $h_I$  are the amplitude of the edge waves on the two droplets. As discussed in section 2.2, the gauge invariance of the electromagnetic field requires  $\nu_I$  to satisfy the sum rule

$$\sum \nu_I = \nu \quad (2.29)$$

(see (2.21) and (2.27)).

Because the electrons are interacting with each other, the edge velocities are determined by  $v_I = cE_I^*/B$  where  $E_I^*$  are the effective electric fields that include both the contributions from the edge potential and the electrons. In order for the Hamiltonian to be bounded from below, we require  $\nu_I v_I > 0$ . We find that the stability of the  $\nu = \frac{2}{5}$  FQH state requires both  $v_I$  to be positive.

Generalizing the discussion in the section 2.1, we find that the electron operators on the two edges are given by

$$\Psi_I = e^{i\frac{1}{\nu_I}\phi_I(x)} \quad I = 1, 2 \quad (2.29)$$

with  $\partial_x \phi_I = \frac{1}{2\pi} \rho_I$ . The electron propagators have the form

$$\langle T(\Psi_I(x, t) \Psi_I^\dagger(0)) \rangle = e^{ik_I x} \frac{1}{(x - v_I t)^{-1/|\nu_I|}} \quad I = 1, 2 \quad (2.30)$$

where  $k_I = r_I/l_B^2$ .

According to the hierarchical picture the  $\nu = \frac{2}{3}$  FQH state is also formed by two condensates, an electron condensate with filling fraction 1 and a hole condensate with filling fraction  $-\frac{1}{3}$ . Thus the above discussion can also be applied to the  $\nu = \frac{2}{3}$  FQH state by choosing  $(\nu_1, \nu_2) = (1, -\frac{1}{3})$ . Again there are two branches of the edge excitations but now with *opposite* velocities if the Hamiltonian is positive definite. This results, although surprising, is not difficult to understand. The stability of both the electron droplet and the hole droplet requires  $E_1^*$  and  $E_2^*$  to have opposite sign.

As we bring two edges together ( $r_1 - r_2 \sim l_B$ ) the interaction between the two branches of the edge excitations can no longer be ignored. In this case the Hamiltonian has a form

$$H = 2\pi \sum_{IJ, k>0} V_{IJ} \rho_{Ik} \rho_{J-k} \quad (2.31)$$

(The Hamiltonian may also contain terms that describe the electron hopping between edges. But those terms are irrelevant at low energies due to the chiral property of the edge excitations. For example one can show that those terms can never open an energy gap<sup>9</sup>) The Hamiltonian (2.31) can still be diagonalized. For  $\nu_1 \nu_2 > 0$  we may choose

$$\begin{aligned} \tilde{\rho}_{1k} &= \cos(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1k} + \sin(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2k} \\ \tilde{\rho}_{2k} &= \cos(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2k} - \sin(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1k} \\ \tan(2\theta) &= 2 \frac{\sqrt{|\nu_1 \nu_2|} V_{12}}{|\nu_1| V_{11} - |\nu_2| V_{22}} \end{aligned} \quad (2.32)$$

One can easily check that

$$\begin{aligned} [\tilde{\rho}_{Ik}, \tilde{\rho}_{Jk'}] &= \frac{\text{sgn}(\nu_I)}{2\pi} k \delta_{IJ} \delta_{k+k'} \\ H &= 2\pi \sum_{I, k>0} \text{sgn}(\nu_I) \tilde{v}_I \tilde{\rho}_{Ik} \tilde{\rho}_{I-k} \end{aligned} \quad (2.33)$$

where the new velocities of the edge excitations  $\tilde{v}_I$  are given by

$$\begin{aligned} \text{sgn}(\nu_1) \tilde{v}_1 &= \frac{\cos^2(\theta)}{\cos(2\theta)} |\nu_1| V_{11} - \frac{\sin^2(\theta)}{\cos(2\theta)} |\nu_2| V_{22} \\ \text{sgn}(\nu_2) \tilde{v}_2 &= \frac{\cos^2(\theta)}{\cos(2\theta)} |\nu_2| V_{22} - \frac{\sin^2(\theta)}{\cos(2\theta)} |\nu_1| V_{11} \end{aligned} \quad (2.34)$$

We see that there are still two branches of the edge excitations. However in this case the edge excitations with a definite velocity are mixtures of those on the inner edge and the outer edge. One can also show that as long as the Hamiltonian (2.31) is bounded from below, the velocities of the two branches  $\tilde{v}_I$  are always positive.

By rewriting electron operator  $\Psi_I$  in terms of  $\tilde{\rho}_I$  we find that their propagators are given by

$$\langle T(\Psi_I(x, t) \Psi_I^\dagger(0)) \rangle = e^{ik_I x} \frac{1}{(x - \tilde{v}_1 t)^{\alpha_I}} \frac{1}{(x - \tilde{v}_2 t)^{\beta_I}} \quad (2.35)$$

where

$$(\alpha_1, \alpha_2) = \left( \frac{1}{|\nu_1|} \cos^2 \theta, \frac{1}{|\nu_2|} \sin^2 \theta \right), \quad (\beta_1, \beta_2) = \left( \frac{1}{|\nu_1|} \sin^2 \theta, \frac{1}{|\nu_2|} \cos^2 \theta \right) \quad (2.36)$$

When the two edges are close to each other within the magnetic length, the  $\Psi_I$  are no longer the most general electron operator on the edge. The generic electron operator may contain charge transfers between the two edges. For the  $\nu = 2/5$  FQH state, the inner edge and the outer edge are separated by the  $\nu = \frac{1}{3}$  Laughlin state. Thus the elementary charge transfer operator is given by

$$\eta(x) = e^{i(\phi_1 - \frac{\nu_1}{\nu_2} \phi_2)} = (\Psi_1 \Psi_2^\dagger)^{\nu_1} \quad (2.37)$$

which transfer  $\nu_1 e = e/3$  charge from outer edge to the inner edge. The generic electron operator then takes the form

$$\begin{aligned} \Psi(x) &= \sum_{n=-\infty}^{+\infty} c_n \psi_n(x) \\ \psi_n(x) &= \Psi_1(x) \eta^n(x) \end{aligned} \quad (2.38)$$

To understand this result, we notice that each operator  $\psi_n$  always create an unit localized charge and is a fermionic operator regardless the value of the integer  $n$ . Therefore each  $\psi_n$  is a candidate for the electron operator on the edge. For a generic interacting system the electron operator on the edge is expected to be a superposition of different  $\psi_n$ 's as represented in (2.38). Note  $\Psi_2 = \psi_{-\frac{1}{\nu_1}}$ . The propagator of  $\psi_n$  can be calculated in the similar way as in section 2.1 and is given by

$$\langle T(\psi_n(x, t) \psi_m^\dagger(0)) \rangle \propto \delta_{n,m} e^{i[k_1 + n\nu_1(k_2 - k_1)]x} \prod_I (x - \tilde{v}_I t)^{-\gamma_{In}} \quad (2.39)$$

Where  $\gamma_{In}$  are

$$\begin{aligned} \gamma_{1n} &= \left[ \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \cos \theta - \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \sin \theta \right]^2 \\ \gamma_{2n} &= \left[ \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \sin \theta + \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \cos \theta \right]^2 \end{aligned} \quad (2.40)$$

From (2.38) and (2.39) we see that the electron propagator has singularities at discrete momenta  $k = k_1 + n\nu_1(k_2 - k_1)$ . It is analogue to the  $k_F, 3k_F, \dots$  singularities of the electron propagator in the interacting 1D electron systems.

For the  $\nu = \frac{2}{3}$  FQH state,  $\nu_1 \nu_2 < 0$ . In this case we need to choose

$$\begin{aligned} \tilde{\rho}_{1k} &= \text{ch}(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1k} + \text{sh}(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2k} \\ \tilde{\rho}_{2k} &= \text{ch}(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2k} + \text{sh}(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1k} \\ \text{th}(2\theta) &= 2 \frac{\sqrt{|\nu_1 \nu_2|} V_{12}}{|\nu_1| V_{11} + |\nu_2| V_{22}} \end{aligned} \quad (2.41)$$

to diagonalize the Hamiltonian. One can check that  $\tilde{\rho}_I$  also satisfies the K-M algebra (2.33) but now

$$\begin{aligned}\operatorname{sgn}(\nu_1)\tilde{v}_1 &= \frac{\operatorname{ch}^2(\theta)}{\operatorname{ch}(2\theta)}|\nu_1|V_{11} - \frac{\operatorname{sh}^2(\theta)}{\operatorname{ch}(2\theta)}|\nu_2|V_{22} \\ \operatorname{sgn}(\nu_2)\tilde{v}_2 &= \frac{\operatorname{ch}^2(\theta)}{\operatorname{ch}(2\theta)}|\nu_2|V_{22} - \frac{\operatorname{sh}^2(\theta)}{\operatorname{ch}(2\theta)}|\nu_1|V_{11}\end{aligned}\tag{2.42}$$

Again as long as the Hamiltonian  $H$  is positive definite, the velocities of the edge excitations  $\tilde{v}_I$  always have opposite signs. The electron operator still has the form (2.38) with  $\eta = \Psi_1\Psi_2^{\dagger\nu_1}$ . The propagator of  $\psi_n$  is still given by (2.39) with

$$\begin{aligned}\gamma_{1n} &= \left[ \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \operatorname{ch}\theta + \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \operatorname{sh}\theta \right]^2 \\ \gamma_{2n} &= \left[ \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \operatorname{sh}\theta + \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \operatorname{ch}\theta \right]^2\end{aligned}\tag{2.43}$$

From (2.40) and (2.43), we see that exponents in the electron propagator depend on the interedge interactions. However we can show that the exponents satisfy a sum rule:

$$\sum_I \operatorname{sgn}(\nu_I)\gamma_{In} \equiv \lambda_n = \left( n + \frac{1}{|\nu_1|} \right)^2 \nu_1 + \frac{n^2 \nu_1^2}{\nu_2}\tag{2.44}$$

$\lambda_n$  always take odd-integer values and are independent of the details of the electron system. The quantization of  $\lambda_n$  is again due to the fact that  $\lambda_n$  are directly linked to the statistics of the electrons:

$$\psi_n(x)\psi_n(x') = (-)^{\lambda_n}\psi_n(x')\psi_n(x)\tag{2.45}$$

We would like to point out that the values of  $\lambda_n$  are determined by the internal correlations (topological orders) of the bulk FQH state.  $\lambda_n$  can be changed only by changing the bulk topological orders through a *two dimensional* phase transition, despite that they are a property of 1D edge excitations. Therefore  $\lambda_n$  are topological quantum numbers that can be used to characterize and to experimentally measure the 2D bulk topological orders.

The above picture of the edge excitations can be confirmed through numerical calculations.<sup>33,34</sup> First we notice that the edge excitations on a circular droplet of a FQH fluid can be labeled by the angular momentum  $\Delta M$  carried by the excitations, where  $\Delta M = M - M_0$ ,  $M$  is the total angular momentum of the excited state and  $M_0$  the angular momentum of the ground state. We also notice that the momentum  $k$  along the edge and the angular momentum  $\Delta M$  are related through  $Rk = \Delta M$ , where  $R$  is the radius of the droplet. If there are two branches of the edge excitations moving in the same direction, one can show, from (2.28), that spectrum of the edge excitation should look like the one in Fig. 3a. For example, the two excited states at  $\Delta M = 1$  are given by  $\rho_{1,k}|0\rangle$  and  $\rho_{2,k}|0\rangle$  with  $k = \frac{1}{R}$ . Such a spectrum has been observed in numerical calculations of the  $\nu = \frac{2}{5}$  FQH state on a circular disc (see Fig. 3b). Fig. 3b is the energy spectrum of a system of 8 electrons in 19 orbits with  $V_1$  interaction. The ground state at  $M = 70$  contain 2 quasiparticles.

If the two edge branches have opposite velocities, the spectrum of the edge excitations should look like the one in Fig. 4a according to (2.28). We find such a spectrum in the

numerical calculation of the  $\nu = \frac{2}{3}$  FQH state (see Fig. 4b). The ground state at  $M = 102$  contain 12 electrons and 4 holes. The similar results have also been obtained in Ref. 33.

We would like to point out that the above results for the  $\nu = 2/3$  FQH state only apply to the sharp edges. By sharp edge we mean the filling fraction drop from the bulk value  $2/3$  to zero within a distance of order magnetic length. For very smooth edge potentials electrons near the edge may form a  $\nu = 1/3$  FQH state between the  $\nu = 2/3$  FQH state and vacuum.<sup>10</sup> In this case the gapless edge excitations are located near two transition regions,  $2/3 \rightarrow 1/3$  and  $1/3 \rightarrow 0$ . There are three branches of edge excitations (one right-moving branch and two left-moving branches) on the boundary between the  $2/3$  and  $1/3$  FQH states and there is one branch of right-moving excitations on the edge of the  $1/3$  state.

## 2.4 The edge excitations of generic FQH states - parton construction

In this section we will use the parton construction to obtain the edge excitations of the generic FQH states.<sup>22</sup> Let us first review the parton construction of the bulk FQH states, using  $\nu = \frac{n}{mn+1}$  FQH state as an example.<sup>14</sup> The trick is to split the electrons into  $m$  charge  $\frac{ne}{mn+1}$  partons  $\psi_\alpha|_{\alpha=1}^m$  and a charge  $\frac{e}{mn+1}$  parton  $\psi_{m+1}$ . All the partons have fermionic statistics and  $m$  even. Since each kind of partons has the same density as the electrons:  $n_\alpha = \psi_\alpha^\dagger \psi_\alpha = n_e$ , the parton  $\psi_{m+1}$  have a filling fraction  $\nu_{m+1} = n$  and the parton  $\psi_\alpha|_{\alpha=1}^m$  have a filling fraction  $\nu_\alpha = 1$ . Thus if the partons were independent, the partons  $\psi_{m+1}$  will form a  $\nu = n$  IQH state described by the wave function  $\chi_n(z_i^{(m+1)})$  and the partons  $\psi_\alpha|_{\alpha=1}^m$  will form  $\nu = 1$  states described by  $\chi_1(z_i^{(\alpha)})$ . But in reality the partons are not independent. By introducing the fictitious particles, the partons, we introduce some unphysical degrees of freedom, namely the density fluctuations of the form  $n_\alpha - n_{\alpha'}$ . In a physical electron states, the densities of the partons always satisfy the constraint

$$\sum_\alpha C_\alpha n_\alpha = 0, \quad \text{for any } \sum_\alpha C_\alpha = 0$$

To use the parton picture to describe the real electron states and to obtain the correct physics for the electrons, we need to make a projection to project away all the unphysical degrees of freedom. The ground state (trial) wave function  $\Psi$  that satisfies the above constraint can be obtained by doing a projection  $z_i^{(1)} = z_i^{(2)} = \dots = z_i$ , where  $z_i$  are the electron coordinates:

$$\Psi(z_i) = \chi_n(z^{(m+1)}) \prod_{\alpha=1}^m \chi_1(z_i^{(\alpha)}) \Big|_{z_i^{(1)} = \dots = z_i^{(m+1)} = z_i} \quad (2.46)$$

The parton construction is very convenient for the construction of the edge excitations. This is because the projection can be done at the effective theory level.

Let us first discuss the edge excitations in the  $\nu = \frac{n}{mn+1}$  FQH state ( $m$  is an even integer). We assume that the FQH state has a disk-like geometry. The edge of the disk is parameterized by  $x$ . We see from the last three sections that the edge excitations are described by the  $U(1)$  K-M algebras. Those edge excitations can be regarded as surface waves propagating along the edge of the incompressible QH fluid. The surface wave can

be described by the ‘‘edge density’’  $\rho(x) = n_e h(x)$  where  $h(x)$  is the displacement of the edge.

Before the projection, all the partons are independent. The charge  $\frac{n}{mn+1} = q_2$  partons  $\psi_\alpha|_{\alpha=1}^m$  form a  $\nu_\alpha = 1$  IQH state and support a single branch of edge excitations for each  $\alpha$ . Those excitations are described by the following K-M algebra

$$[\rho_{\alpha,k}, \rho_{\alpha',k'}] = \frac{k}{2\pi} \delta_{k+k'} \delta_{\alpha\alpha'} \quad (2.47)$$

where  $\rho_\alpha$  is the edge density of the  $\alpha^{\text{th}}$  partons. The parton creation operator on the edge is given by  $\psi_\alpha = e^{i\phi_\alpha}$  for  $\alpha = 1, \dots, m$ , where  $\partial_x \phi_\alpha = 2\pi\rho_\alpha$ . The charge  $\frac{1}{mn+1} = q_1$  partons  $\psi_{m+1}$  form a  $\nu = n$  IQH states that support  $n$  branches of edge excitations. Those edge excitations are described by

$$[\rho_{i,k}, \rho_{i',k'}] = \frac{k}{2\pi} \delta_{k+k'} \delta_{i,i'}, \quad i, i' = 1, 2, \dots, n \quad (2.48)$$

where  $\rho_i$  is the edge density of the partons  $\psi_{m+1}$  in the  $i^{\text{th}}$  Landau level. The parton creation operators on the edge are given by  $\psi_i = e^{i\phi_i}$  with  $\partial_x \phi_i = 2\pi\rho_i$ ,  $i = 1, \dots, n$ . They carry an electric charge  $q_1$ .  $\psi_i$  are the partons in the  $i$ th Landau level. The coupling between the edge densities and the external electric potential is given by

$$e(q_2 \sum_{\alpha=1}^m \rho_\alpha + q_1 \sum_{i=1}^n \rho_i) A_0 \quad (2.49)$$

Before the projection, the Hilbert space of the edge excitations is generated by  $\rho_\alpha$ ,  $\rho_i$ ,  $\psi_\alpha$  and  $\psi_i$  with  $\alpha = 1, \dots, m$  and  $i = 1, \dots, n$ , which contains  $n + m$  branches.

Because the fluctuations associated with  $\tilde{\rho}_C = \sum_{\alpha=1}^m C_\alpha \rho_\alpha - C_{m+1} \sum_{i=1}^n \rho_i$  are unphysical for any  $C_a$  satisfying  $\sum_{a=1}^{m+1} C_a = 0$ , we should remove all such fluctuations to obtain the correct edge excitations for electrons. To accomplish this, we will first specify the physical operators. A physical operator must not create any fluctuations associated with  $\tilde{\rho}_C$ . Hence a physical operator must commute with  $\tilde{\rho}_C$ :

$$[\hat{O}_{phy}, \tilde{\rho}_C] = 0 \quad (2.50)$$

for any  $C_a$  that satisfy  $\sum_{a=1}^{m+1} C_a = 0$ . One can easily check that the following edge density operators are physical

$$\begin{aligned} j_0 &= \sqrt{\nu} (\rho_0 + \frac{1}{n} \sum_{i=1}^n \rho_i) \\ j_i &= \sum_{j=1}^n a_i^j \rho_j, \quad i = 1, \dots, n-1 \end{aligned} \quad (2.51)$$

where  $\rho_0 = \sum_{\alpha=1}^m \rho_\alpha$ .  $a_i^j$  in (2.51) are orthogonal vectors satisfy  $\sum_{j=1}^n a_i^j = 0$  and  $\sum_{j=1}^n a_i^j a_{i'}^j = \delta_{i,i'}$ . Similarly the charged physical operators (with the minimum charge) are given by  $\Psi_i = e^{i(\phi_i + \sum_{\alpha=1}^m \phi_\alpha)}$ ,  $i = 1, \dots, n$ . The operators  $\Psi_i$  carry an electric charge

*e.* They are just the electron creation operators on the edge. The Hilbert space of the physical edge excitations is generated by  $j_i$  and  $\Psi_i$ , thus contains  $n$  branches. We find that the edge excitations of the  $\nu = \frac{n}{mn+1}$  FQH state (obtained from the parton construction) have  $n$  branches.

We would like to remark that there is a gauge symmetry in the above construction. The gauge symmetry is generated by  $\tilde{\rho}_C$ . (2.50) is just the gauge invariant condition of the physical operators. The appearance of the gauge symmetry is due to the introduction of the unphysical degrees of freedom.

From (2.51), (2.47) and (2.48) we see that the physical edge density operators satisfy the following K-M algebra

$$[j_{i,k}, j_{i',k'}] = \frac{k}{2\pi} \delta_{k+k'} \delta_{i,i'} \quad i = 0, \dots, n-1 \quad (2.52)$$

From (2.49) we find that only  $j_0$  couples to the electric potential  $e\sqrt{\nu}j_0A_0$ .

Using the algebra (2.47) and (2.48) we can easily calculate the equal time correlations between  $\Psi_i$  and  $\Psi_i^\dagger$ :  $\langle \Psi_i^\dagger(x) \Psi_j(y) \rangle \propto (x-y)^{-m-1} \delta_{i,j}$ . The electronic state on the edge is definitely not a Fermi liquid due to the anomalous exponent in the correlation functions. We can also show that  $\{\Psi_i(x), \Psi_j(y)\} = 0$ . Therefore  $\Psi_i$  are indeed fermionic operators. We will discuss the charged excitations created by  $\Psi_i$  in more detail in chapter 3.

When  $n = m = 2$ , (2.46) describes a  $\nu = 2/5$  FQH state. After a proper redefinition of  $\rho_i$ , one can show that edge excitations (together with the electron operators) described by (2.52) and (2.28) are identical. This result strongly suggests that the  $\nu = 2/5$  FQH states obtained from the standard hierarchical construction and from the parton construction belong to the same universality class.

The above results can be easily generalized to the FQH state described by the following wave function<sup>14</sup>

$$\Psi(z_i) = \prod_{i=1}^p \chi_{n_i}(z_i) \quad (2.53)$$

where  $p$  is odd. The filling fraction of the above state is  $\nu = (\sum_{i=1}^p \frac{1}{n_i})^{-1}$ . The edge excitations can be shown to have  $1 + \sum_i (n_i - 1)$  branches. The electron operators on the edge have the following propagator

$$\langle \Psi^\dagger(x) \Psi(y) \rangle \propto \left(\frac{1}{x-y}\right)^{l+p} \quad (2.54)$$

Notice that when  $n_i = 1$ , (2.53) becomes the Laughlin wave functions. In this case the above results reduce to the results obtained in Ref. 24 (see also the section 2.1). Clearly the construction also applies to even more general hierarchy FQH states. For example  $\Psi_{n_i}$  in (2.53) does not have to be an IQH wave function. It can be a FQH wave function with a filling fraction  $\nu = 1 \pm \frac{1}{l}$ .

We would like to remark that the above result is correct only when all non-unit  $n_i$  are not equal to each other. If several  $n_i$  are equal to an integer that is not equal to 1, there will be a non-abelian symmetry between partons. In this case the edge states are described by non-abelian K-M algebras. (See Ref. 19 and chapter 5.)

## 2.5 Effective theory and the edge states

In this section we will directly derive the macroscopic theory of the edge excitations from the effective theory of the bulk FQH states.<sup>15,30</sup> In this approach we do not rely on a specify construction of the FQH states. The relation between the bulk topological orders and edge states also becomes clear in this approach.

We know a hierarchical (or generalized hierarchical) FQH state contains many different condensates, the electron condensate form the Laughlin states and the quasiparticle condensate on top of that form a hierarchical state. Each condensate correspond to one component of the incompressible fluid. The idea is to generalize the hydrodynamical approach in the section 2.1 to multi-component fluids and to obtain the low energy effective theory of the edge excitations. To accomplish this, we first would like to write down the low energy effective theory of the bulk FQH state. The effective theory should contain the information about the topological orders in the bulk states.

The different condensates in the FQH states are not independent. The particles in one condensate behave like a flux tube to the particles in other condensates. To describe such a coupling, it is convenient to use  $U(1)$  gauge fields to describe the density and the current of each condensate. In this case the couplings between different condensates are described by Chern-Simons term of the gauge fields.<sup>35,15,17,21</sup> By some further consideration of the electron operators in the effective theory, it was shown<sup>21</sup> that the most general abelian FQH states of the electrons are classified by integer valued symmetry matrix  $K$  with odd diagonal elements and are described by the following effective theory<sup>36</sup>

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I \partial a_J + a_I \cdot j_I + \frac{1}{2\pi} A \partial a_I + g_{IJ} f_I \cdot f_J \quad (2.55)$$

where  $a_I \partial a_J$  is a short hand notation for  $a_{I\mu} \partial_\nu a_{J\lambda} \epsilon_{\mu\nu\lambda}$ ,  $\mu, \nu, \lambda = 0, 1, 2$  and  $f_{I\alpha\beta}$  is field strength of the  $U(1)$  gauge field  $a_{I\mu}$ . The FQH state described by (2.55) contain  $\kappa$  different condensates and there are  $\kappa$  kinds of different quasiparticle excitations, where  $\kappa$  is the rank of  $K$ .  $j_{I\mu}$  is the density and the current of the  $I^{th}$  kind of excited quasiparticles (denoted by  $\psi_I$ ) that behaves like vortices in the condensates.  $j_{I\mu}$  are normalized such that  $\int dx j_{I0}$  are integers. The density and the current of the  $I^{th}$  component of incompressible fluid (*i.e.*, the  $I^{th}$  condensate) is given by

$$J_{I\mu} = \frac{1}{4\pi} \epsilon_{\mu\alpha\beta} f_{I\alpha\beta} \quad (2.56)$$

The filling fraction of the FQH state is

$$\nu = \sum_{IJ} (K^{-1})_{IJ} \quad (2.57)$$

As we create an  $J^{th}$  quasiparticle  $\psi_J$ , it will induce a change in the density of the  $I^{th}$  condensates,  $\delta J_{I0}$ . From the equation of motion we find that  $\delta J_{I0}$  satisfy

$$\int d^2x \delta J_{I0} = (K^{-1})_{JI} \quad (2.58)$$

(Note  $\int d^2x j_{J0} = 1$  in presence of the  $J^{th}$  quasiparticle.) The charge and the statistics of the  $J^{th}$  quasiparticle is given by

$$\theta_J = \pi(K^{-1})_{JJ}, \quad q_J = \sum_I (K^{-1})_{JI} \quad (2.59)$$

A generic electron excitation can be written as a bound state of the quasiparticles:

$$j_e = \sum_{I,J} L_I K_{IJ} j_J \quad (2.60)$$

where  $L_I$  are integers satisfying  $\sum_I L_I = 1$ . From (2.4.2) we can easily show that electron excitations in (2.60) satisfy the following properties: A) They carry an unit charge (see (2.59). B) They have the fermionic statistics. C) Moving an electron excitation defined in (2.60) around any quasiparticle excitations always induces a phase of multiple of  $2\pi$ . D) The excitations defined in (2.60) are all the excitations satisfying the above three condition. For a more detailed discussion, see Ref. 21.

We would like to point out that the effective theory (2.55) not only applies to the standard QH system in which all electrons are spin polarized and in the first Landau level, it also applies to the QH system in which electrons may occupy several Landau levels and/or occupy several layers and/or carry different spins. In this case the index  $I$  may label the condensates in different Landau levels, in different layers and/or with different spins. The edge excitations for the spin 1/2 electron system was discussed in Ref. 9,37.

To understand the relation between the effective theory and the edge states, let us first consider the simplest FQH state of the filling fraction  $\nu = 1/q$  and try to rederive the results in section 2.1 from the bulk effective theory. Such a FQH state is described by  $U(1)$  Chern-Simons theory with the action:<sup>13,30</sup>

$$S = \frac{q}{4\pi} \int a \partial a d^3x \quad (2.61)$$

Suppose that our sample has a boundary. For the simplicity we shall assume that the boundary is the  $x$ -axis and the sample is the lower half-plane. The Chern-Simons action is not invariant under gauge transformations  $a_\mu \rightarrow a_\mu + \partial_\mu f$  due to the boundary effects. To solve this problem we will restrict the gauge transformations to be zero on the boundary  $f(x, y=0) = 0$ . Due to this restriction some degrees of freedom of  $a_\mu$  on the boundary become dynamical. We know the effective theory (2.61) is derived only for a bulk FQH state without boundary. We will take (2.61) with the restricted gauge transformation as the definition of the effective theory for a FQH state with boundary. Such a definition is definitely self consistent. In the following we will show that such a definition reproduces the results obtained in section 2.1.

One way to study the dynamics of a gauge theory is to choose the gauge condition  $a_0 = 0$  and regard the equation of motion for  $a_0$  as a constraint. For the Chern-Simons theory such a constraint becomes  $f_{ij} = 0$ . Thus we write  $a_i$  as  $a_i = \partial_i \phi$ . Plug this into (2.61), one obtains<sup>38</sup> an effective conformal theory on the edge with an action

$$S_{edge} = \frac{m}{4\pi} \int \partial_t \phi \partial_x \phi dx dt \quad (2.62)$$

This approach, however has a setback. It is easy to see that a Hamiltonian associated with the action (2.62) is zero and the boundary excitations described by eq. (2.62) have zero velocity. Therefore this action cannot be used to describe any physical edge excitations connected with the FQHE. The edge excitations in the FQH states always have finite velocities.

The appearance of finite velocities of edge excitations is a boundary effect. The bulk effective theory defined by eq. (2.55) does not contain the information about the velocities of the edge excitations. The edge velocities in the QH states are actually determined by the edge potentials. To determine the dynamics of the edge excitations from the effective theory we must find a way to input the information about the edge velocity. The edge velocities must be treated as the external parameters that are not contained in the bulk effective theory. The problem is how to put in these parameters in the theory.

Let us now note that the condition  $a_0 = 0$  is not a unique choice of the gauge fixing condition. More general gauge fixing condition has a form

$$a_\tau = a_0 + va_x = 0 \quad (2.63)$$

Here  $a_x$  are the component of the vector potential parallel to the boundary of the sample and  $v$  is a parameter that has a dimension of velocity.

It is convenient to choose new coordinates that satisfy

$$\begin{aligned} \tilde{x} &= x - vt \\ \tilde{t} &= t, \quad \tilde{y} = y \end{aligned} \quad (2.64)$$

In these coordinates the components of the gauge field are given by

$$\begin{aligned} \tilde{a}_{\tilde{t}} &= a_t + va_x \\ \tilde{a}_{\tilde{x}} &= a_x \\ \tilde{a}_{\tilde{y}} &= a_y \end{aligned} \quad (2.65)$$

The gauge fixing condition becomes the one discussed before. It is easy to see that the form of the Chern-Simons action is preserved in the new coordinates:

$$S = \frac{q}{4\pi} \int d^3x a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} = \frac{q}{4\pi} \int d^3x \tilde{a}_{\tilde{\mu}} \partial_{\tilde{\nu}} \tilde{a}_{\tilde{\lambda}} \epsilon^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} \quad (2.66)$$

Repeating the previous derivation, we find the edge action is given by

$$S = \frac{q}{4\pi} \int d\tilde{t} d\tilde{x} \partial_{\tilde{t}} \phi \partial_{\tilde{x}} \phi \quad (2.67)$$

In terms of the original physical coordinates the above action acquires a form

$$S = \frac{q}{4\pi} \int dt dx (\partial_t + v\partial_x) \phi \partial_x \phi \quad (2.68)$$

which is a chiral boson theory. It is easy to see that the edge excitation described by (2.68) have a non-zero velocity. The quantization of chiral boson theory has been discussed in

detail in Ref. 39. The canonical momentum  $\pi(x)$  is equal to  $\pi = \frac{\delta L}{\delta \dot{\phi}} = \frac{q}{4\pi} \partial_x \phi$ . The coordinate  $\phi$  and momentum  $\pi$  obey the commutation relations:

$$\begin{aligned} [\pi(x), \phi(y)] &= \frac{1}{2} \delta(x - y) \\ [\phi(x), \phi(y)] &= \frac{\pi}{q} \text{sgn}(x - y) \end{aligned} \quad (2.69)$$

The Hamiltonian of the theory (2.68) is given by

$$H = -\frac{qv}{4\pi} \int dx \partial_x \phi \partial_x \phi \quad (2.70)$$

The Hilbert space contains only left-moving degrees of freedom (or right moving degrees of freedom if  $v < 0$ ). The theory (2.69) and (2.70) describes free left (or right) moving phonons (*i.e.*, the edge density waves). One can easily show that (2.69) and (2.70) are equivalent to the K-M algebra (2.7) by identifying  $\rho = \frac{1}{2\pi} \partial_x \phi$ .

In the following we would like to show that  $\rho = \frac{1}{2\pi} \partial_x \phi$  can really be interpreted as the 1D electron density on the edge. First we notice that the coupling between the electrons and the external electromagnetic potential is given by  $\int A_\mu J_\mu d^3x = \int \frac{1}{2\pi} A \partial a d^3x$  (see (2.56)). From  $\tilde{a}_{\vec{i}} = \partial_{\vec{i}} \phi$  we see that

$$\int A_\mu J_\mu d^3x = \int d\tilde{x} d\tilde{t} \frac{1}{2\pi} A_{\tilde{t}} \partial_{\tilde{x}} \phi = \int dx dt \frac{1}{2\pi} (A_t + v A_x) \partial_x \phi \quad (2.71)$$

where we have used the equation of motion  $(\partial_t + v \partial_x) \phi = 0$  and the transformation (2.64) and (2.65). (2.71) clearly indicates that the 1D edge electron density is given by  $\frac{1}{2\pi} \partial_x \phi = \rho$ .

The velocity of the edge excitations  $v$  enters our theory through the gauge fixing condition. Notice that under the restricted gauge transformations the gauge fixing conditions (2.63) with different  $v$  cannot be transformed into each other. They are physically inequivalent. This agrees with our result that  $v$  in the gauge fixing condition is physical and actually determines the velocity of the edge excitations.

The Hamiltonian (2.70) is bounded from below only when  $vq < 0$ . The consistency of our theory requires  $v$  and  $q$  to have opposite signs. Therefore the sign of the velocity (the chirality) of the edge excitations is determined by the sign of the coefficient in front of the Chern-Simons terms.

The above results can be easily generalized to the generic FQH states described by (2.55) because the matrix  $K$  can be diagonalized. The resulting effective edge theory has a form

$$S_{edge} = \frac{1}{4\pi} \int dt dx [K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J] \quad (2.72)$$

The Hamiltonian is given by

$$H_{edge} = \frac{1}{4\pi} \int dt dx V_{IJ} \partial_x \phi_I \partial_x \phi_J \quad (2.73)$$

Therefore  $V$  must be a positive definite matrix. Using this result one can show that a positive eigenvalue of  $K$  corresponds to a left moving branch and a negative eigenvalue corresponds to a right moving one.

The effective theory of the  $\nu = 2/5$  FQH state is given by<sup>21</sup>

$$K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \quad (2.74)$$

Since  $K$  have two positive eigenvalues, the edge excitations of the  $\nu = 2/5$  FQH state have two branches moving in the same direction. The  $\nu = 1 - \frac{1}{n}$  FQH state is described by the effective theory with

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -n \end{pmatrix} \quad (2.75)$$

The two eigenvalues of  $K$  now have opposite sign, hence the two branches of the edge excitations move in opposite directions. This prediction was suggested in Ref. 40 and has been confirmed by numerical calculations.<sup>33,34</sup>

### 3. CHARGED EXCITATIONS AND ELECTRON PROPAGATOR ON THE EDGES OF GENERIC FQH STATES

In the last chapter we studied dynamics of the edge excitations of the FQH effects at low energies. We found that the low lying edge excitations are described by a free phonon theory that is exactly soluble. In this chapter we will concentrated on the generic charge excitations. In particular we will calculate the propagators of the electrons and the quasiparticles for the most general (abelian) FQH state discussed in Ref. 21 and in chapter 2.4. The key point is again to write the electron or the quasiparticle operators in terms of the phonon operator  $\rho_I$ . Once we do so, the propagators can be easily calculated because the phonons are free (at low energies and long wave length).

We know for a FQH state described by (2.55), the edge states are described by the action (2.72). The Hilbert space of the edge excitations forms a representation of K-M algebra

$$[\rho_{Ik}, \rho_{Jk'}] = (K^{-1})_{IJ} \frac{1}{2\pi} k \delta_{k+k'} \quad (3.1)$$

$$k, k' = \text{integer} \times \frac{2\pi}{L}$$

where  $\rho_I = \frac{1}{2\pi} \partial_x \phi_I$  is the edge density of the  $I^{\text{th}}$  condensate in the FQH state,  $I, J = 1, \dots, \kappa$ , and  $\kappa$  is the dimension of  $K$ . The electron density on the edge is given by

$$\rho_e = \sum_I \rho_I \quad (3.2)$$

The dynamics of the edge excitations are described by the Hamiltonian:

$$H = 2\pi \sum_{IJ} V_{IJ} \rho_{I,k} \rho_{J,-k} \quad (3.3)$$

where  $V_{IJ}$  is a positive definite matrix.

Let us first try to write down the  $I_0^{th}$  quasiparticle operator  $\psi_{I_0}$ . We know that inserting the quasiparticle on the edge will cause a change  $\delta\rho_I$  in the edge density of the  $I^{th}$  condensate (see (2.58)) that satisfies

$$\int dx \delta\rho_I = (K^{-1})_{I_0 I}. \quad (3.4)$$

Because  $\psi_{I_0}$  is a local operator that only causes a local change of the density, we have

$$[\rho_I(x), \psi_{I_0}(x')] = (K^{-1})_{I_0 I} \delta(x - x') \psi_{I_0}(x') \quad (3.5)$$

Using the Kac-Moody algebra (3.1) one can show that the quasiparticle operators that satisfy (3.5) are given by

$$\psi_{I_0} \propto e^{i\phi_{I_0}} \quad (3.6)$$

The charge of the quasiparticle  $\psi_{I_0}$  is determined from the commutator  $[\rho_e, \psi_{I_0}]$  and is given by

$$q_{I_0} = \sum_I (K^{-1})_{II_0} \quad (3.7)$$

We know that any excitations in the FQH states (include the electrons) can be constructed from the fundamental quasiparticle excitations  $\psi_I$  discussed above. From (2.60) we see that the electron operator can be written as

$$\begin{aligned} \Psi_{e,L} &\propto \prod_I \psi_I^{l_I} \propto e^{i \sum_I l_I \phi_I} \\ l_I &= \sum_J K_{IJ} L_J \\ \sum_I L_I &= 1 \end{aligned} \quad (3.8)$$

The above operators carry unit charge as one can see from (3.7). The commutation of the  $\Psi_{e,L}$  can be calculated as

$$\begin{aligned} \Psi_{e,L}(x) \Psi_{e,L}(x') &= (-)^\lambda \Psi_{e,L}(x') \Psi_{e,L}(x) \\ \lambda &= \sum_{IJ} L_I K_{IJ} L_J \end{aligned} \quad (3.9)$$

Because the diagonal elements of  $K$  are odd integers, we can show that  $(-)^{\lambda} = -1$ . This is the expected result for electron operators.

Since all the operators  $\Psi_{e,L}$  for different choice of  $L_I$  carry unit charge and are fermionic, so each  $\Psi_{e,L}$  can be a candidate for the electron operators. In general the true electron operator is a superposition of  $\Psi_{e,L}$ 's:

$$\Psi_e = \sum_L C_L \Psi_{e,L} \quad (3.10)$$

In this paper when we say there are many different electron operators on the edge, we really mean that the true physical electron operator is a superposition of the those operators.

Using the K-M algebra (3.1) and the Hamiltonian (3.3), we can calculate the propagators of generic quasiparticle operator

$$\Psi_l \propto e^{i \sum_I l_I \phi_I} \quad (3.11)$$

(which include the electron operators for suitable choice of  $l$ ). First we notice that after a suitable redefinition of  $\rho_I$ :

$$\tilde{\rho}_I = \sum_J U_{IJ} \rho_J \quad (3.12)$$

$K$  and  $V$  can be simultaneously diagonalized, *i.e.*, in terms of  $\tilde{\rho}_I$  (3.1) and (3.3) become

$$\begin{aligned} [\tilde{\rho}_{Ik}, \tilde{\rho}_{Jk'}] &= \sigma_I \delta_{IJ} \frac{1}{2\pi} k \delta_{k+k'} \\ H &= 2\pi \sum_I |v_I| \tilde{\rho}_{I,k} \tilde{\rho}_{I,-k} \end{aligned} \quad (3.13)$$

where  $\sigma_I = \pm 1$  is the sign of the eigenvalues of  $K$ . The velocity of the edge excitations created by  $\tilde{\rho}_I$  is given by  $v_I = \sigma_I |v_I|$ .

To prove the above result we first redefine  $\rho_I$  to transform  $V$  into the identity matrix:  $V \rightarrow U_1 V U_1^T = 1$ . This is possible because  $V$  is a positive definite symmetric matrix. Now  $K$  becomes a new symmetric matrix  $K_1 = U_1 K U_1^T$  whose eigenvalues have the same sign as the eigenvalues of  $K$  (although the absolute values may differ). Then we make an orthogonal transformation to diagonalize  $K_1$ :  $(K_1)_{IJ} \rightarrow \sigma_I |v_I| \delta_{IJ}$ . After a trivial rescaling of the densities, we obtain (3.13). In terms of  $\tilde{\rho}_I$  the operators  $\Psi_l$  has a form

$$\begin{aligned} \Psi_l &\propto e^{i \sum_I \tilde{l}_I \tilde{\phi}_I} \\ \tilde{l}_J &= \sum_I l_I U_{IJ}^{-1} \end{aligned} \quad (3.14)$$

Form (3.13) and (3.14) we see that the propagator of  $\Psi_l$  has the following general form

$$\langle \Psi_l^\dagger(x, t) \Psi_l(0) \rangle \propto e^{i l_I k_I x} \prod_I (x - v_I t + i \sigma_I \delta)^{-\gamma_I}, \quad \gamma_I = \tilde{l}_I^2 \quad (3.15)$$

where  $v_I = \sigma_I |v_I|$  is the velocities of the edge excitations and  $k_I$  is momentum of the  $I^{th}$  quasiparticle  $\psi_I$  on the edge.  $\gamma_I$  in (3.15) satisfy the sum rule

$$\sum_I \sigma_I \gamma_I \equiv \lambda_l = \sum_I l_I K_{IJ}^{-1} l_J \quad (3.16)$$

In order to prove the above sum rule, we have used the relation

$$(U K U^T)_{IJ} = \sigma_I \delta_{IJ} \quad (3.17)$$

From (3.9) and (3.16) we see that the sum rule is directly related to the statistics of  $\Psi_l$  and  $\lambda_l$  is a topological quantum number. If  $\Psi_l$  represent the electron operator,  $\lambda_l$  will be an odd integer. From (3.15) we also see that the operator  $\Psi_l$  create an excitation with momentum near  $\sum_I l_I k_I$ .

Now let discuss the hierarchical states with filling fractions  $\nu = \frac{p}{pq+1}$  ( $q = \text{even}$ ) in more detail. Those states include  $\nu = 2/5, 3/7, 2/9, \dots$  FQH states. The hierarchical states with  $\nu = \frac{p}{pq+1}$  is described by  $p$  by  $p$  matrices  $K = 1 + qC$ , where  $C$  is the pseudo identity matrix:  $C_{IJ} = 1, I, J = 1, \dots, p$ . We have  $K^{-1} = 1 - \frac{q}{pq+1}C$ . Because all the edge excitations move in the same direction, we have

$$\langle \Psi_l^\dagger(x=0, t) \Psi_l(0) \rangle \propto t^{-\lambda_l} \quad (3.18)$$

where  $\lambda_l$  is given by (3.16). The fundamental quasiparticle is given by  $l^T = (1, 0, \dots, 0)$  that carries charge  $\frac{1}{pq+1}$ . The exponent in its propagator is  $\lambda_l = 1 - \frac{q}{pq+1}$ . The quasiparticle with the smallest exponent is given by  $l^T = (1, \dots, 1)$  that carries charge  $\frac{p}{pq+1}$ . The exponent is  $\lambda_l = \frac{p}{pq+1}$  that is less than  $1 - \frac{q}{pq+1}$  (note we have  $q \geq 2$  and  $p \geq 1$ ). Later in section 6.2, we will see that such a charge  $\frac{p}{pq+1}$  quasiparticle dominate the tunneling between two edges of the *same* FQH fluid at low energies.

The electron operators are given by  $\Psi_{e,L}$  with  $l$  satisfying  $\sum_I l_I = pq+1$ . The exponent in the propagator is given by  $\lambda_l = \sum l_I^2 - q(pq+1)$ . The electron operator with minimum exponent in its propagator is given by  $l = (q, \dots, q, q+1)$ . The value of the minimum exponent is  $\lambda_l = q+1$ . Such an electron operator dominate the tunneling between edges of two *different* FQH fluid at low energies.

In section 2.5 we argued that the edge excitations of the a FQH state characterized by a matrix  $K$  are described by the  $U(1)$  K-M algebras characterized by the same matrix (see (3.1)). This result is correct only for sharp edges. For the smooth edges the filling fraction may not drop to zero directly near the edge, but through several intermediate FQH states. Such a composite edge in general contain more branches of edge excitations. However the new edge branches always appear in pairs (one left moving and one right moving). The edge excitations on a smooth edge are in general described by the following matrix:

$$K_{edge} = K_{bulk} \oplus \begin{pmatrix} K' & 0 \\ 0 & -K' \end{pmatrix}$$

where  $K_{bulk}$  is the matrix characterizing the bulk FQH state and  $K'$  is another integer matrix that depend on the edge potential. For the  $2/3$  edge ( $2/3 \rightarrow 1/3 \rightarrow 0$ ) discussed at the end of section 2.3, the edge excitations are described by

$$K_{edge} = \begin{pmatrix} -3 & & & \\ & 1 & & \\ & & -3 & \\ & & & 3 \end{pmatrix}$$

#### IV. MICROSCOPIC THEORY OF THE EDGE EXCITATIONS OF THE LAUGHLIN STATE

In the last two chapters we developed the effective low energy theory of the edge excitations for the most generic abelian FQH states. In this chapter we will present a microscopic theory for the edge excitations in the Laughlin states. Haldane pointed out that the edge excitations of the Laughlin states can be generated by multiplying symmetric polynomials to the Laughlin wave functions.<sup>26</sup> Stone studied the structures of the Hilbert space for the  $\nu = 1$  IQH state using the symmetric polynomials.<sup>27</sup> In the following we will generalize Stone's results to the Laughlin states.

To be specific, let us consider an electron gas in first Landau level. We choose  $V(\vec{r}) \propto \partial^2 \delta(\vec{r})$  as the interactions between electrons. Because A)  $H_V = \sum_{ij} V(\vec{r}_i - \vec{r}_j)$  is positive definite (*i.e.*,  $\langle \psi | H_V | \psi \rangle \geq 0$  for any  $|\psi\rangle$ ) and B)  $H_V$  have zero expectation value when all pairs of electron have relative angular momentum  $m \geq 3$ , thus the  $\nu = 1/3$  Laughlin wave function

$$\Phi_3(z_i) = Z^{-1/2} \prod_{i < j} (z_i - z_j)^3 \prod_k e^{-\frac{1}{4}|z_k|^2} \quad (4.1)$$

has zero energy and is an exact ground state of our Hamiltonian. In (4.1)  $Z$  is the normalization factor. However the Laughlin state (4.1) is not the only state with zero energy. One can easily check that the following type of states all have zero energy:

$$\Phi(z_i) = P(z_i) \Phi_3(z_i) \quad (4.2)$$

where  $P(z_i)$  is a symmetric polynomial of  $z_i$ . In the fact the reverse is also true: all the zero energy states are of form (4.2). This is because in order for a fermion state to have zero energy,  $\Phi$  must vanishes at least as fast as  $(z_i - z_j)^3$  when any two electrons  $i$  and  $j$  are brought together (the possibility  $(z_i - z_j)^2$  is excluded by the fermion statistics). Because the Laughlin wave function is zero only when  $z_i = z_j$  therefore  $P = \Phi/\Phi_3$  is a finite function. Since  $\Phi$  and  $\Phi_3$  are both antisymmetric functions in the first Landau level,  $P$  is symmetric holomorphic function that can only be a symmetric polynomial.

Among all the states in (4.2), the Laughlin state describe a circular droplet with smallest radius. All other states are deformation and/or inflation of the droplet of the Laughlin state. Thus the states generated by  $P$  correspond to the edge excitations of the Laughlin state.

Now let us first consider the zero energy space (*i.e.*, the space of symmetric polynomials). We know the space of symmetric polynomials is generated by the following polynomials  $s_n = \sum_i z_i^n$  (through multiplication and addition). Let  $M_0 = 3 \frac{N(N-1)}{2}$  be the total angular momentum of the Laughlin state (4.1). Then the state  $\Phi$  will have an angular momentum  $M = \Delta M + M_0$  where  $\Delta M$  is the order of the symmetric polynomial  $P$ . Since we have only one order-zero and order-one symmetric polynomial  $s_0 = 1$  and  $s_1 = \sum_i z_i$ , thus the zero energy states for  $\Delta M = 0, 1$  are non-degenerate. However when  $\Delta M = 2$  we have two zero energy states generated by  $P = s_2$  and  $P = s_1^2$ . For general  $\Delta M$  the degeneracy of the zero energy states is given by

$$\begin{array}{l} \Delta M : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\ \text{degeneracy :} \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 7 \quad 11 \end{array} \quad (4.3)$$

Here we would like to point out that the degeneracy in (4.3) is exactly what we expected from the macroscopic theory. We know for a circular droplet, the angular momentum  $\Delta M$  can be regarded as the momentum along the edge  $k = 2\pi\Delta M/L$  where  $L$  is the perimeter of the QH droplet. According to the macroscopic theory the (neutral) edge excitations are generated by the density operators  $\rho_k$ . One can easily check that the edge states generated by the density operators have same degeneracies as those in (4.3) for every  $\Delta M$ . *e.g.*, the two state at  $\Delta M = 2$  is generated by  $\rho_{\kappa_0}^2$  and  $\rho_{2\kappa_0}$  where  $\kappa_0 = 2\pi/L$ . Therefore the space generated by the K-M algebra (2.7) and the space of the symmetric polynomials are identical.

Now let us ask a physical question. Are the symmetric polynomials generate all the low energy states? If this is true, then from the above discussion we see that all the low energy excitations of the HQ droplet are generated by the K-M algebra and we can say that (2.7) is a complete theory of the low lying excitations. Unfortunately up to now we do not have an analytic proof of the above statement. This is because although states orthogonal to the states generated by the symmetric polynomials have non-zero energies, it is not clear that those energies remain finite in the thermodynamical limit. It is possible that the energy gap approaches to zero in the thermodynamical limit. To resolve this problem, right now we have to rely on numerical calculations. In Fig. 5 we present the energy spectrum of a system of six electron in the first 22 orbits for the Hamiltonian introduced at the beginning of this chapter. The degeneracies of the zero energy states at  $M = 45, \dots, 51$  (or  $\Delta M = 0, \dots, 6$ ) are found to be 1, 1, 2, 3, 5, 7, 11, which agrees with (4.3). More importantly we see clearly a finite energy gap separate all other states from the zero energy states. Thus the numerical results imply that all the low lying edge excitations of the Laughlin state are generated by the symmetric polynomials or the K-M algebra (2.7).

In the following we are going to derive the relation between the generators of the symmetric polynomials,  $s_n$ , and the generators of the K-M algebra,  $\rho_{n\kappa_0}$ . To find the relation between those operators, we need to study the inner-product in the two Hilbert spaces. The inner-product in the space generated by the K-M algebra is known. In the following we will use the plasma analogue to calculate the inner-product in the space generated by the symmetric polynomials.

As we will see later that it is more convenient to study the edge states generated by

$$\psi(z_i; \xi) = \prod_i \left(1 - \frac{z_i}{\xi}\right) \quad (4.4)$$

the symmetric function  $\psi(\xi)$  is related to the generators  $s_n$  through

$$\psi(z_i; \xi) = \exp\left(-\sum_{n>0} \frac{1}{n} \xi^{-n} s_n\right) \quad (4.5)$$

First we would like to calculate the norm of the state generated by  $\psi(\xi)$

$$\begin{aligned} \langle \psi^n(\xi) | \psi^n(\xi) \rangle &= |\xi|^{-2Nn} \frac{Z_1}{Z} \\ Z &= \int \prod d^2 z_i \exp\left(\sum_{ij} 2m \ln |z_i - z_j| - \sum_k \frac{1}{2} |z_k|^2\right) \\ Z_1 &= \int \prod d^2 z_i \exp\left[\sum_{ij} 2m \ln |z_i - z_j| + \sum_k \left(-\frac{1}{2} |z_k|^2 + 2n \ln |\xi - z_k|\right)\right] \end{aligned} \quad (4.6)$$

where  $m = 1/\nu$  for the  $\nu = 1/m$  Laughlin state and  $|\psi^n(\xi)\rangle = \psi^n(z_i, \xi)\Phi_m(z_i)$ . Notice that  $Z$  is the partition function of one component plasma, and  $Z_1$  is the partition function of the plasma interacting with a charge at  $\xi$ . If we ignore the discreteness of the charges and treat the plasma as a continuous medium, we may write

$$Z = e^E, \quad Z_1 = e^{E_1} \quad (4.7)$$

where  $E$  and  $E_1$  are the total energies of the plasma. We expect the above approximation gives rise to the right ratio  $Z_1/Z$  if  $\xi$  is not so close to the droplet, *i.e.*,  $|\xi| - R \gg 1$  where  $R = \sqrt{mN}$  is the radius of the droplet.

The plasma behave like a metal. Thus the change in energy as we add an external charge is given by

$$E_1 - E = 2nN \ln |\xi| - \frac{n^2}{m} \left[ \ln\left(|\xi| - \frac{R^2}{|\xi|}\right) - \ln(|\xi|) \right] + O(N^{-1}) \quad (4.8)$$

We know the external charge change the shape of the droplet. The first term in (4.8) is the interaction between the external charge and the *undeformed* droplet. The second term is the correction due to the deformation of the droplet. This correction can be represented by the interaction of the external charge with its mirror images. We notice that  $E_1 - E \rightarrow 2nN \ln |\xi|$  as  $\xi \rightarrow \infty$ . From (4.8) we find the norm of  $|\psi^n(\xi)\rangle$  to be

$$\langle \psi^n(\xi) | \psi^n(\xi) \rangle = \left( \frac{\xi \xi^*}{\xi \xi^* - R^2} \right)^{n^2/m} \quad (4.9)$$

Because the inner-product  $\langle \psi^n(\tilde{\xi}) | \psi^n(\xi) \rangle$  is a holomorphic function of  $\xi$  and an anti-holomorphic function of  $\tilde{\xi}$ , thus (4.9) implies that

$$\langle \psi^n(\tilde{\xi}) | \psi^n(\xi) \rangle = \left( \frac{\xi \tilde{\xi}^*}{\xi \tilde{\xi}^* - R^2} \right)^{n^2/m} \quad (4.10)$$

We can also show a more general results using a similar calculation

$$\langle \prod_i \psi^{n_i}(\tilde{\xi}_i) | \prod_i \psi^{n_i}(\xi_i) \rangle = \prod_{i,j} \left( \frac{\xi_i \tilde{\xi}_j^*}{\xi_i \tilde{\xi}_j^* - R^2} \right)^{n_i n_j / m} \quad (4.11)$$

Now let us consider the following operator in the K-M algebra

$$\psi_{KM}(\xi) = \exp\left(-\sum_{n>0} \frac{1}{n} \sqrt{2\pi} R^{n+\frac{1}{2}} \rho_{n\kappa_0} \xi^{-n}\right) \quad (4.12)$$

where  $\kappa_0 = 1/R$ . Using the K-M algebra we can easily show that the states generated by  $\psi_{KM}$  also have an inner-product of the form given by (4.11):

$$\langle 0 | \prod_i \psi_{KM}^{\dagger n_i}(\tilde{\xi}_i) \prod_i \psi_{KM}^{n_i}(\xi_i) | 0 \rangle = \prod_{i,j} \left( \frac{\xi_i \tilde{\xi}_j^*}{\xi_i \tilde{\xi}_j^* - R^2} \right)^{n_i n_j / m} \quad (4.13)$$

Because both the operators  $\psi(\xi)$  and  $\psi_{KM}(\xi)$  generate the whole Hilbert spaces and because (4.11) and (4.13) imply that  $\psi(\xi)$  and  $\psi_{KM}(\xi)$  have the same matrix elements, we conclude that the symmetric functions  $\psi(\xi)$  correspond to the operator  $\psi_{KM}(\xi)$  in the K-M algebra. From (4.5) and (4.12) we find the following simple relation between the two sets of generators:

$$s_n = \sqrt{2\pi} R^{n+\frac{1}{2}} \rho_k, \quad k = n\kappa_0 = \frac{n}{R} \quad (4.14)$$

Notice that

$$\xi^{Nm} \tilde{\xi}^{*Nm} \langle \psi^m(\tilde{\xi}) | \psi^m(\xi) \rangle = \xi^{Nm} \tilde{\xi}^{*Nm} \left( \frac{\xi \tilde{\xi}^*}{\xi \tilde{\xi}^* - R^2} \right)^m \quad (4.15)$$

is proportional to the electron propagator  $G_e$  along the edge of a  $N_e = N + 1$  electron system at equal time. Choosing  $\xi = R e^{i2\pi x/L}$  and  $\tilde{\xi} = R$ , we get

$$G_e(x) = L^{-m} a^{m-1} e^{im(N_e - \frac{1}{2}) \frac{2\pi x}{L}} \sin^{-m}(\pi x/L) \quad (4.16)$$

that reduces to (2.12) when  $x$  is much less than  $L$ . Here  $a$  is a length scale of order  $l_B$ . (4.16) can be expanded:

$$\begin{aligned} G_e(x) &= L^{-m} a^{m-1} e^{im(N_e - 1) \frac{2\pi x}{L}} \left[ \sum_{n=0}^{\infty} e^{-i \frac{2\pi x}{L} n} \right]^m \\ &= L^{-m} a^{m-1} e^{im(N_e - 1) \frac{2\pi x}{L}} \sum_{n=0}^{\infty} C_{m+n-1}^n e^{-i \frac{2\pi x}{L} n} \\ C_{m+n-1}^n &= \frac{(n+m-1)!}{(m-1)!n!} \end{aligned} \quad (4.17)$$

From this expansion we obtain the electron occupation number  $n_M$  at the angular momentum  $M$  state:

$$\begin{aligned} n_M &= 0, \quad M > m(N_e - 1) \\ n_M &= \frac{a^{m-1}}{L^{m-1}} C_{mN_e - M - 1}^{m(N_e - 1) - M}, \quad M \leq m(N_e - 1) \end{aligned} \quad (4.18)$$

We see that exact position of the Fermi edge is at the last partially occupied single-particle orbit, *i.e.*, at angular momentum  $m(N_e - 1)$  (or  $k_F = \sqrt{m(N_e - 1)}/l_B$ ). Note when  $m(N_e - 1) - M \gg m$ ,  $n_M \propto (m(N_e - 1) - M)^{m-1}$ . Or in terms of momentum along the edge, we have  $n_k \propto (k_F - k)^{m-1}$  and  $n_k$  does not have a jump at the Fermi momentum. The electron occupation number  $n_M$  has been studied numerically in Ref. 41,33,42.

We would like to remark that (4.16) is correct only when  $x$  is much larger than the magnetic length  $l_B$ . Therefore (4.18) is valid only when  $m(N_e - 1) - M \ll \sqrt{N_e}$ .

If the dispersion of the edge excitations is linear,  $G_e$  can only depend on  $x - vt$  at low energies. We immediately see that

$$G_e(x, t) = L^{-m} a^{m-1} e^{im(N - \frac{1}{2}) \frac{2\pi(x-vt)}{L}} \sin^{-m}[\pi(x - vt)/L] \quad (4.19)$$

The result (4.16) is also supported by a numerical calculation. In Fig. 6 we present a numerical result of the electron propagator on the edge for the  $\nu = \frac{1}{2}$  Laughlin state of 36 bosons. The solid line is the theoretical result (4.16).

In summary we develop the microscopic theory of the edge excitations of the Laughlin states. From the above discussion, one can see that the Hilbert space (together with the inner-product) of the edge excitations is identical with the Hilbert space of the K-M algebra. The operators in the two Hilbert spaces are related in a simple way (see (4.14)). The electron occupation number  $n_M$  near the edge was also calculated exactly.

## 5. EDGE EXCITATIONS OF NON-ABELIAN QH STATES

It is now known that there are two classes of QH states, abelian QH states whose quasiparticles all have abelian statistics and non-abelian QH states<sup>18,19,20</sup> that contain some quasiparticles with non-abelian statistics.<sup>43</sup> In the following we will briefly study the edge excitations of some non-abelian QH states.

A class of non-abelian QH states is represented by wave functions<sup>19</sup>

$$\chi_1^p(z_i)[\chi_m(z_i)]^n \quad (5.1)$$

whose filling fraction is  $\nu = (p + \frac{m}{n})^{-1}$ . In (5.1),  $p + n$  is an odd integer and  $\chi_m$  is the fermion wave function with  $m$  filled Landau levels. The electrons in this wave function stay within the first  $n(m - 1) + 1$  Landau levels. There are local Hamiltonians such that (5.1) is an exact ground state.

The non-abelian QH state represented by (5.1) contain a quasiparticle excitation with charge  $\frac{n}{pnm+m^2}$ . Such a quasiparticle can be shown to carry a non-abelian statistics described by the  $SU(n)$  level  $m$ , or in short  $SU(n)_m$ , Chern-Simons theory. For more detailed discussions see Ref. 19 and 20.

We can also view the Landau level indices as the layer indices and write the related wave functions for multilayered electron systems.<sup>20</sup> More general non-abelian states were discussed in Ref. 20. In general the non-abelian QH states are most likely to appear in multilayer systems and/or in systems with small energy gaps between Landau levels. The simplest non-abelian FQH state is represented by the wave function  $\chi_1\chi_2^2$  that has a filling fraction  $\nu = \frac{1}{2}$ . Such a non-abelian FQH state may appear in three-layer electron systems. A different class of non-abelian QH states was suggested in Ref. 18.

To understand the edge states of the non-abelian QH states, let us first concentrate on the non-abelian state described by  $[\chi_m]^n$ . Note such a state can be obtained through the parton construction<sup>14</sup> by “splitting” electrons into  $n$  charge  $1/n$  partons. Therefore it is convenient to use the method discussed in section 2.4 to construct the edge states of non-abelian states. Let us use  $\alpha = 1, \dots, n$  to label different partons. We first assume that all partons are independent, and thus the ground state wave function is given by

$$\prod_{\alpha=1}^n \chi_m(z_i^{(\alpha)}) \quad (5.2)$$

where  $z_i^{(\alpha)}$  is the coordinates of the  $\alpha^{th}$  kind of the partons. The original electron wave function  $[\chi_m(z_i)]^n$  is obtained by doing the projection  $z_i^{(1)} = \dots = z_i^{(n)} = z_i$ . Before the projection the state (5.2) is just  $n$  independent IQH states each with filling fraction  $m$  and contain  $mn$  branches of edge excitations that is described by the following low energy effective theory in 1D:

$$\mathcal{L} = \sum_{\alpha,a} i\psi^{\alpha a \dagger}(x,t)(\partial_t - v\partial_x)\psi^{\alpha a}(x,t) \quad (5.3)$$

where  $\psi^{\alpha a}$ ,  $a = 1, \dots, m$ , are charge  $1/n$  fermion fields that describe the  $a^{th}$  edge branch of the  $\alpha^{th}$  kind of the partons. (Note that each kind of the partons forms the  $\nu = m$  IQH state that contains  $m$  branches of edge excitations.)

Within the effective theory (5.3) we can define three densities  $J(x) = \frac{1}{n} \sum_{\alpha a} \psi^{\alpha a \dagger}(x)\psi^{\alpha a}(x)$ ,  $j^s(x) = \sum_{\alpha,a,b} \psi^{\alpha a \dagger}(x)T_{ab}^s \psi^{\alpha b}(x)$  and  $\tilde{j}^{\tilde{s}}(x) = \sum_{\alpha,\beta,a} \psi^{\alpha a \dagger}(x)\tilde{T}_{\alpha\beta}^{\tilde{s}} \psi^{\beta a}(x)$ , where  $T_{ab}^s$  ( $\tilde{T}_{\alpha\beta}^{\tilde{s}}$ ) is generators of the  $SU(m)$  ( $SU(n)$ ) Lie algebra.  $J$  is the electric charge density of the edge excitations and  $j^s$  and  $\tilde{j}^{\tilde{s}}$  represent the densities of non-abelian charges associated with the Landau levels and the partons respectively. The above three densities satisfy the  $U(1) \times SU(m)_n \times SU(n)_m$  K-M algebra:<sup>44</sup>

$$\begin{aligned} [J_k, J_{k'}] &= \frac{m}{2\pi n} k \delta_{k+k'} \\ [j_k^r, j_{k'}^s] &= \frac{n}{4\pi} k \delta_{r,s} \delta_{k+k'} + L^{-\frac{1}{2}} f^{rst} j_{k+k'}^t \\ [\tilde{j}_k^{\tilde{r}}, \tilde{j}_{k'}^{\tilde{s}}] &= \frac{m}{4\pi} k \delta_{\tilde{r},\tilde{s}} \delta_{k+k'} + L^{-\frac{1}{2}} \tilde{f}^{\tilde{r}\tilde{s}t} \tilde{j}_{k+k'}^t \end{aligned} \quad (5.4)$$

where  $f^{rst}$  ( $\tilde{f}^{\tilde{r}\tilde{s}t}$ ) is the structure constant of the  $SU(m)$  ( $SU(n)$ ) Lie algebra.  $J$ ,  $j$  and  $\tilde{j}$  commute with each other. It has been shown that the excitations in (5.3) are completely described by the above K-M algebra.<sup>44</sup>

Now we are ready to do the projection and to discuss the edge excitations of the electron system. We know the independent parton model contains unphysical degrees of freedom. To remove the unphysical fluctuation, we need to recombine the partons into electrons. The electron operator  $\psi_e = \prod \psi^\alpha$  is a  $SU(n)$  singlet under the  $SU(n)$  transformation  $\psi^\alpha \rightarrow U^{\alpha\beta} \psi^\beta$ . (Here  $\psi^\alpha$  is the operator of the  $\alpha^{th}$  kind of the partons.) Therefore any excitations of the electron system must be  $SU(n)$  singlets and  $\tilde{j}^{\tilde{s}}$  are identically zero for physical excitations. Therefore we can recombine partons into electron by removing all excitations with non-trivial  $SU(n)$  quantum numbers. A physical operator must not create any  $SU(n)$  fluctuations and therefore must commute with the  $SU(n)$  density operator  $\tilde{j}$ :

$$[\hat{O}_{phy}, \tilde{j}^{\tilde{s}}] = 0 \quad (5.5)$$

The above equation is the analogue of the equation (2.50) in the section 2.4.

We can see that all the operators in the  $U(1)$  and the  $SU(m)_n$  K-M algebra commute with  $\tilde{j}$ , therefore edge excitations of the electronic state  $(\chi_m)^n$  are described by the  $U(1) \times SU(m)_n$  K-M algebra.

The additional factor  $\chi_1^p$  in the electron wave function (5.1) will only modify the  $U(1)$  K-M algebra. One can show that the edge excitations in the state (5.1) are described by the following  $U(1) \times SU(m)_n$  K-M algebra:

$$\begin{aligned} [J_k, J_{k'}] &= \frac{\nu}{2\pi} k \delta_{k+k'} \\ [j_k^r, j_{k'}^s] &= \frac{n}{2\pi} k \delta_{k+k'} + f^{rst} j_{k+k'}^t \\ \nu &= \frac{m}{pm + n} \end{aligned} \quad (5.6)$$

More precisely, the Hilbert space of the low energy edge excitations forms a representation of the above algebra. The low energy effective Hamiltonian takes the following general form

$$H = \sum_k V J_k J_{-k} + \sum_{k;r,s} V_{r,s} j_k^r j_{-k}^s + \sum_{k;s} V_s (j_k^s J_{-k} + J_k j_{-k}^s) \quad (5.7)$$

(5.6) and (5.7) give us a complete description of the low energy dynamics of the edge excitations of non-abelian QH states. Note that although the Hilbert space is generated by the  $U(1) \times SU(m)_n$  K-M algebra, the Hamiltonian in general does not respect the  $SU(m)$  symmetry. So when we say the edge excitations of a non-abelian state are described by non-abelian K-M algebra, we only mean that the Hilbert space of the low lying excitations forms a representation of the non-abelian algebra.

Now let us study the electron operators and the quasiparticle operators. First let us assume  $p = 0$ . From the parton construction and requirement that the electron operators must satisfy (5.5), we find that the electron operators on the edge are given by

$$\begin{aligned} \Psi_e^M &= \epsilon_{\alpha_1, \dots, \alpha_n} \psi^{\alpha_1 a_1} \dots \psi^{\alpha_n a_n} S_{a_1, \dots, a_n}^M \\ &= e^{i \frac{n}{m} \phi} V_{\Lambda_n, M} \end{aligned} \quad (5.8)$$

where  $\epsilon_{\alpha_1, \dots, \alpha_n}$  are rank  $n$  antisymmetric tensors (labeled by  $M$ ),  $S_{a_1, \dots, a_n}^M$  is the rank  $n$  symmetric tensor, and  $V_{\Lambda_n, M}$  is the primary field of the  $SU(m)_n$  K-M algebra in the representation of the rank  $n$  symmetric tensor. The field  $\phi$  in (5.8) is determined through  $J(x) = \frac{1}{2\pi} \partial_x \phi(x)$  and  $e^{i \frac{n}{m} \phi}$  is a primary field of the  $U(1)$  K-M algebra. The electron operators in (5.8) satisfy (5.5) and are  $SU(n)$  singlet. They belong to the  $U(1) \times SU(m)_n$  K-M algebra and can be written as a product of the primary fields in the  $U(1)$  and the  $SU(m)_n$  K-M algebra.<sup>45</sup> Note that there are many different electron operators just as what happens in the IQH states (with  $\nu > 1$ ) and the hierarchical FQH states. Those different electron operators form a representation of the rank  $n$  symmetric tensors of the  $SU(m)$  group. (5.8) also implies  $\Psi_e^M$  anti-commute (commute) when  $n$  is odd (even). The equal time correlation has a form

$$\langle \Psi_e^{\dagger M}(x) \Psi_e^M(y) \rangle = \langle e^{-i \frac{n}{m} \phi(x)} V_{\Lambda_n, M}^{\dagger}(x) e^{i \frac{n}{m} \phi(y)} V_{\Lambda_n, M}(y) \rangle \propto \left( \frac{1}{x-y} \right)^n \quad (5.9)$$

When  $p \neq 0$ , we only need to modify the  $U(1)$  vertex operator. We find in this case the electron operators are given by

$$\Psi_e^M = e^{i \frac{1}{\nu} \phi} V_{\Lambda_n, M} \quad (5.10)$$

where  $\nu$  is given in (5.6). Using (5.6) one can easily check that  $\Psi_e^M$  in (5.10) create a unit localized charge. We can also show that

$$\frac{e^{i\frac{1}{\nu}\phi(x)}e^{i\frac{1}{\nu}\phi(y)}}{e^{i\frac{n}{m}\phi(x)}e^{i\frac{n}{m}\phi(y)}} = (-)^p \frac{e^{i\frac{1}{\nu}\phi(y)}e^{i\frac{1}{\nu}\phi(x)}}{e^{i\frac{n}{m}\phi(y)}e^{i\frac{n}{m}\phi(x)}} \quad (5.11)$$

Therefore  $\Psi_e^M$  anti-commute when  $n + p = \text{odd}$ . In this case  $\Psi_e^M$  is a fermionic operator as expected. The equal time correlation is given by

$$\langle \Psi_e^{\dagger M}(x)\Psi_e^M(y) \rangle = \langle e^{-i\frac{1}{\nu}\phi(x)}V_{\Lambda_n, M}^{\dagger}(x)e^{i\frac{1}{\nu}\phi(y)}V_{\Lambda_n, M}(y) \rangle \propto \left(\frac{1}{x-y}\right)^{n+p} \quad (5.12)$$

The exponent  $n + p$  in the electron propagator can be measured experimentally through tunneling between edge states.<sup>23</sup> (See section 6.2 for a more detailed discussion.) Such a measurement combining with the knowledge of the filling fraction may allow us to determine experimentally whether a FQH state is an abelian or a non-abelian state.

Now let us consider the quasiparticle operators. A generic quasiparticle operator is a product of a primary field in the  $U(1)$  K-M algebra and a primary field  $V_{\Lambda}$  in the  $SU(m)$  K-M algebra:

$$\Psi_q = e^{iq_{\Lambda}\phi}V_{\Lambda} \quad (5.13)$$

where  $\Lambda$  labels the representation of the primary field. The value of the  $q_{\Lambda}$  is determined by requiring the electron wave function in presence of the quasiparticle to be single valued. A calculation of  $q_{\Lambda}$  can be found in Ref. 20. The electric charge of the quasiparticle is given by

$$Q = q_{\Lambda}\nu \quad (5.14)$$

Within the edge theory, the single-valueness of the electron wave function is equivalent to the requirement that the electron operators  $\Psi_e$  and the quasiparticle operator  $\Psi_q$  are mutually local to each other. This means the correlation function

$$\langle T(\Psi_e(x_1, t_1)\Psi_q(x_2, t_2)\dots) \rangle \quad (5.15)$$

to be single valued as the electron operator goes around the quasiparticle operator in the space-time. In (5.15) “...” represents other operators that make the correlation function non-zero.

In the following we will attempt to calculate  $q_{\Lambda}$  within the edge theory. For simplicity let us consider the quasiparticles in the fundamental representation (labeled by  $\Lambda_1$ ) of the  $SU(m)$  and assume the Hamiltonian to respect the  $SU(m)$  symmetry, *i.e.*,  $V_{rs} \propto \delta_{rs}$  and  $V_r = 0$  in (5.7). In this case we can use the conformal field theory to calculate the correlations. The primary fields  $V_{\Lambda_1}$  and  $V_{\Lambda_n}$  satisfy the following operator product expansion:

$$V_{\Lambda_1}(z_1)V_{\Lambda_n}(z_2) \propto (z_1 - z_2)^{h_0 - h_1 - h_n}V_{\Lambda_0} \quad (5.16)$$

where  $h_0$ ,  $h_1$  and  $h_n$  are the conformal dimensions of the primary fields  $V_{\Lambda_0}$ ,  $V_{\Lambda_1}$  and  $V_{\Lambda_n}$  respectively.  $z$  in (5.16) is given by  $x + i\tau$  and  $\tau$  is the imaginary time *it*. One can show that<sup>46</sup>

$$h_0 - h_1 - h_n = \frac{1}{m} \quad (5.17)$$

From the operator product expansion

$$e^{i\frac{1}{\nu}\phi(z_1)}e^{iq_{\Lambda_1}\phi(z_2)} = (z_1 - z_2)^\lambda e^{i(\frac{1}{\nu}+q_{\Lambda_1})\phi(z_2)}$$

$$\lambda = \frac{\nu}{2}\left[\left(\frac{1}{\nu} + q_{\Lambda_1}\right)^2 - \frac{1}{\nu^2} - q_{\Lambda_1}^2\right] = q_{\Lambda_1} \quad (5.17)$$

we see that the single valuedness of the correlation requires that  $\lambda + h_0 - h_1 - h_n = \text{integer}$ , or

$$q_{\Lambda_1} = -\frac{1}{m} + l \quad (5.18)$$

where  $l$  is an integer. We find the quasiparticle operator has a form

$$\Psi_q^{(l)} = e^{i(l-\frac{1}{m})\phi}V_{\Lambda_1} \quad (5.19)$$

Such a quasiparticle operator carries a charge  $(l - \frac{1}{m})\nu$ . Thus the quasiparticle mentioned at the beginning of this section correspond to  $\Psi_q^{(0)}$ . The above quasiparticle operator has a conformal dimension  $h = \frac{\nu}{2}(l - \frac{1}{m})^2 + \frac{m^2-1}{2m(m+n)}$ .<sup>46</sup> Thus the propagator of the quasiparticle is given by

$$\langle \Psi_q^{(l)\dagger}(x, t)\Psi_q^{(l)}(0) \rangle \propto (x - v_J t)^{-\nu(l-\frac{1}{m})^2} (x - v_j t)^{-\frac{m^2-1}{m(m+n)}} \quad (5.20)$$

where  $v_J$  and  $v_j$  are the velocities of the excitations in the  $U(1)$  and the  $SU(m)_n$  K-M algebras respectively. In some sense the quasiparticles described by (5.19) are fundamental. All other quasiparticles can be constructed as bound states of the quasiparticles in (5.19).

In real samples we do not have the  $SU(m)$  symmetry and different excitations in the  $SU(m)$  K-M algebra do not move with the same velocity  $v_j$ . In this case we can still say some thing about the quasiparticle propagator. First we notice that the asymptotic form of the equal-time correlation can be obtained from the operator product expansion

$$\Psi_q^{(l)\dagger}(x_1, t)\Psi_q^{(l)}(x_2, t) \propto (x_1 - x_2)^{-\nu(l-\frac{1}{m})^2 - \frac{m^2-1}{m(m+n)}} = (x_1 - x_2)^{-g}$$

$$g = \frac{m}{m+n} + \frac{1-p}{(pm+n)(m+n)} + \frac{l^2m-2l}{pm+n} \quad (5.21)$$

and is independent of the Hamiltonian. If the edge excitations still have linear dispersion relation (possibly with many different velocities), then the ‘‘equal-space’’ correlation will has the form

$$\Psi_q^{(l)\dagger}(x, t_1)\Psi_q^{(l)}(x, t_2) \propto (t_1 - t_2)^{-g} \quad (5.22)$$

The exponent in (5.22) can be directly measured in edge tunneling experiments (see Ref. 23 and section 6.2). The quasiparticle with  $l = 0$  has the smallest exponent and may dominate the edge tunneling at low energies.

The non-abelian FQH state also contains a charge  $\nu e$  quasiparticle with abelian statistics  $\theta = \nu\pi$ . Such a quasiparticle is created by the  $U(1)$  vertex operator

$$\Psi_a = e^{i\phi} \quad (5.23)$$

on the edge. This quasiparticle has the following edge propagator

$$\langle \Psi_a^\dagger(x, t)\Psi_a(0, t) \rangle \sim x^{-\nu} = x^{-\frac{m}{pm+n}}. \quad (5.24)$$

Before ending this section we would like to say a few words about the relation between the above low energy effective theory and microscopic wave functions of the edge excitations of the non-abelian QH states. If we treat the partons as the independent particles, it is clear that edge excitations above the independent-parton ground state (5.2) contain  $mn$  branches and described by the  $[U(1)]^{mn}$  K-M algebra. The wave functions of the edge excitations can be written down easily since each kind of partons forms a  $\nu = m$  IQH state. However the orthogonal edge excited states when we treat  $z_i^{(\alpha)}$ ,  $\alpha = 1, \dots, n$ , as independent variables may no longer be orthogonal when we treat  $z_i^{(\alpha)}$  as the same variable  $z_i$ . Therefore the projection  $z_i^{(\alpha)} = z_i$  reduces the dimension of the Hilbert space and changes the structure of the edge states. From the above discussions, we expect the new Hilbert space obtained after the projection is described by the  $U(1) \times SU(m)_n$  K-M algebra.

## 6. APPLICATIONS TO EXPERIMENTS

In this chapter we will apply the theory of the edge excitations to some realistic situations and discuss some experimental predictions. Here we only consider the simplest cases among vast possible experimental applications.

### 6.1 Static edge transport in absence of inter-edge interactions and inter-edge tunneling

In this section we will discuss the static transport properties of large QH samples in which the inter-edge interaction and inter-edge tunneling can be ignored. We will also assume the sample is large enough so that all different branches on the same edge are in equilibrium. In this case we can get some universal results about the transport in the QH samples. The results for non-equilibrium situations depend on the detail of the electron interaction, edge potentials and many other things and have to be discussed for each individual experimental situation.

The problem in this section has been addressed in Ref. 10,11 using Landauer-Buttiker type formalism. Here we will take a different approach which lead to some new results. We find that under certain conditions, the static equilibrium transport properties of the large QH samples do not depend on the internal topological orders in the QH states. The transport properties can be completely determined by the filling fractions.

Let us consider a QH sample that may contain several domains with different filling fractions. We will use  $i$  to label different edges (including the boundary of two neighboring domains). We choose a reference state in which the voltages (*i.e.*, the chemical potentials of the electrons) on all the different edges are equal and are chosen to be zero. In this reference state we have persistent edge currents circulating in the sample. As we change the voltages on the edges, the edge currents will change. Let us use  $I_i$  to denote the change of the edge current on the  $i^{th}$  edge relative to the reference state. Certainly  $I_i$ 's are the function of the edge voltages denoted by  $V_i$ .

In absence of the inter-edge tunneling the edge voltages are constant along each segment of the edges. However the edge voltage may have discontinuous jump at branching points

of the edges. The edge voltages  $V_i$  and (the changes of) the edge currents  $I_i$  are related by a simple thermodynamical relation:

$$I_i = \Delta\nu_i \frac{e^2}{h} V_i \quad (6.1)$$

where  $\Delta\nu_i$  is the difference of the filling fractions on the two side of the  $i^{\text{th}}$  edge.

The branching point of the edges can be classified into eight different types as illustrated in Fig. 7. There, each line represents *all* edge branches moving in the direction indicated by the arrow. Thus a single line may represent several edge branches moving in the same direction. For the vertices in Fig. 7b–d, 7f–h, we need to know the voltages on two edges in order to determine the voltage on the third edge. The voltage on the third edge is determined by current conservation and (6.1):

$$V_1(\nu_a - \nu_b) + V_2(\nu_b - \nu_c) + V_3(\nu_c - \nu_a) = 0 \quad (6.2)$$

The more detailed relations between the three voltages depend on the details of the electron interaction and the edge potential near the branching point. However the vertices in Fig. 7a and 7e are special. The voltages on all the three edges are equal

$$V_1 = V_2 = V_3 \quad (6.3)$$

Now we would like to try to use (6.2) and (6.3) to calculate the static transport properties of some HQ systems. Let us consider a simple QH sample as shown in Fig. 8. We want to determine the voltages on different edge from the total current passing through the Hall bar,  $I = I_1 - I_2 = \dots = I_7 - I_8$ . Certainly we can only determine the voltages up to an overall constant.

If all the vertices are of types in Fig. 7b–d, 7f–h (this is often the case when some edges contain both right and left movers), we can only use (6.2) to determine the voltages. One can easily see that we do not have enough equations to calculate all the unknowns. This implies that the voltages depend on the details how the edge branches reach equilibrium near each vertex and the edge voltages may be different from sample to sample. (However as we will see later if the edge Hamiltonians take some special forms, the voltage on all edges can be uniquely determined from the filling fractions even when some edges contain both right and left movers.)

If every edge only contains excitations moving in one direction, then half the vertices are of the type in Fig. 7a. In this case we have enough equations to determine all the voltages (up to an overall constant). Two situations,  $\nu > \nu_g$  and  $\nu < \nu_g$ , need to be treated separately. When  $\nu > \nu_g$  (*e.g.*,  $\nu = 4/3$ ,  $\nu_g = 1$ ) we find that (Fig. 8)

$$V_1 = V_3 = V_4 = \nu_g^{-1} I \frac{h}{e^2}, \quad V_5 = V_6 = V_8 = 0, \quad V_7 = \nu^{-1} I \frac{h}{e^2}, \quad V_2 = (\nu_g^{-1} - \nu^{-1}) I \frac{h}{e^2} \quad (6.4)$$

The overall constant is fixed by setting  $V_8 = 0$ . When  $\nu < \nu_g$  we find that (Fig. 8)

$$V_2 = V_3 = V_5 = (\nu^{-1} - \nu_g^{-1}) I \frac{h}{e^2}, \quad V_4 = V_6 = V_7 = \nu^{-1} I \frac{h}{e^2}, \quad V_8 = 0, \quad V_1 = (2\nu^{-1} - \nu_g^{-1}) I \frac{h}{e^2} \quad (6.5)$$

The results (6.4) and (6.5) also apply to more general cases (*i.e.*, the cases in which some edges may contain both right and left movers) *under the following condition*: the

interaction between edge excitations is such that all edge branches that couple the electromagnetic field (*i.e.*, with non-zero  $\tilde{\eta}_I$ ) move in the same direction. In this case the edge excitations moving in the opposite directions are neutral excitations and can be ignored. This reduces the problem to the one discussed in the last paragraph.

The above situation can be realized when the Hamiltonian has a form (for the system with two branches of edge excitations)

$$H = \int dx [V_1(\rho_1 + \rho_2)^2 + V_2(\nu_1^{-1}\rho_1 - \nu_2^{-1}\rho_2)^2] \quad (6.6)$$

The branch described by  $\sum_I \rho_I$  exhausts all the coupling to the electromagnetic field. All other branches are decoupled from electromagnetic field. Such a structure of the edge states has been found in real samples in the edge magnetoplasmon experiments where only a single resonant peak is observed.<sup>47</sup> Therefore we expect (6.4) and (6.5) to apply to those samples.

The result (6.4) and (6.5) has been obtained for IQH states by using the Landauer-Buttiker type formalism. (6.4) is generalized to the FQH regime in Ref. 10,11. However the effects of the excitations moving in opposite directions have not been considered. We know (6.4) and (6.5) are in general not correct. It is correct only when all the charged edge excitations are moving in the same direction. However edge magnetoplasma experiments suggest that, for typical samples, there is only one branch of edge excitations that couple to electromagnetic field even for hierarchical FQH states. This implies that (6.4) and (6.5) also apply to those typical samples. This result is a little disappointing, because it implies that the static transport properties of large QH samples do not reveal any internal structures (or the topological orders) of the QH states.

## 6.2 Tunneling between edge states

One of most remarkable properties of the edge states in the FQH effects is that the electrons on the edges are strongly correlated that gives rise to electron propagators with unusual exponents. Those exponents are new quantum numbers that can be measured in edge tunneling experiments,<sup>23</sup> so that we can distinguish different QH states with the same filling fraction.

First let us study the tunneling between the edges of two *different* bulk FQH fluids (see Fig. 9). We will limit ourselves to the tunneling at low voltages and low temperatures so that the retardation of the tunneling can be ignored. (We will not consider the resonant tunneling.<sup>6</sup>) We will also assume that the electron interaction to be short ranged and that the excitations on different edges do not interact with each other. (The coulomb interaction can be screened by a metal gate near the 2D electron gas)

Let us consider a simple case where the tunneling take place only at  $x = 0$ . The tunneling operator now can be written as  $\Gamma A = \Gamma c_L c_R^\dagger|_{x=0}$ , where  $c_{R,L}$  are the electron operators on the edge R and L. The tunneling current between the two edges  $I_t$  is given by the formula<sup>48</sup>

$$I_t(t) = e\Gamma^2 \int_{-\infty}^{+\infty} dt' \theta(t-t') \times \left[ e^{i \int_t^{t'} eV_t(\tilde{t})d\tilde{t}} \langle [A(t), A^\dagger(t')] \rangle - e^{-i \int_t^{t'} eV_t(\tilde{t})d\tilde{t}} \langle [A^\dagger(t), A(t')] \rangle \right] \quad (6.7)$$

where  $V_t$  is the voltage difference between the two edges. Introducing  $f(\omega, t)$  through

$$e^i \int_t^{t'} eV_t(\tilde{t}) d\tilde{t} = \int d\omega f(\omega, t) e^{i(t-t')\omega} \quad (6.8)$$

we find that (6.7) can be rewritten as

$$I_t(t) = -2e\Gamma^2 \int d\omega \text{Im}[f(\omega, t)X_{ret}(-\omega)] \quad (6.9)$$

where  $X_{ret}(\omega)$  is the Fourier transformation of  $X_{ret}(t)$ :

$$X_{ret}(t) = -i\theta(t)\langle[A(t), A^\dagger(0)]\rangle \quad (6.10)$$

For a DC voltage, (6.9) is simplified to

$$I_t = -2e\Gamma^2 \text{Im}[X_{ret}(-eV_t)] \quad (6.11)$$

The formula (6.7) reduces the tunneling problem to a calculation of the correlation function of the tunneling operator  $A$ . In absence of the interactions between the two edges, the correlation function of  $A$  is simply the product of the electron propagators on the two edges. The electron propagators for various FQH states were calculated in the previous chapters.

In general the electron propagator has a form

$$G(x=0, t) = \frac{a^{-1}\omega_0^{-g}}{t^g} \quad (6.12)$$

at equal space point, where  $a$  is the cut-off length scale and  $\omega_0$  is the cut-off frequency scale. The propagator of the tunneling operator  $A$  can be calculated from (6.12)

$$\begin{aligned} G_+(t) &= \langle A(t)A^\dagger(0) \rangle = a^{-2}[-\omega_0^2(t-i\delta)^2]^{-g} \\ G_-(t) &= \langle A^\dagger(0)A(t) \rangle = a^{-2}[-\omega_0^2(t+i\delta)^2]^{-g} \end{aligned} \quad (6.13)$$

The retarded Green function  $X_{ret}$  is given by

$$\begin{aligned} X_{ret}(\omega) &= \int -i\theta(t)[G_+(t) - G_-(t)]e^{i\omega t} dt \\ &= a^{-2}\omega_0^{-2g}|\omega|^{2g-1} \frac{\pi}{\Gamma(2g)} [\text{tg}\pi g + i\text{sgn}(\omega)] \end{aligned} \quad (6.14)$$

where  $\Gamma$  is the Gamma function. When  $g$  is an odd integer  $q$ , (6.14) can be further simplified to

$$X_{ret}(\omega) = -i \frac{\pi}{(2q-1)!} a^{2q-2} v^{-2q} \omega^{2q-1} \quad (6.15)$$

(6.14) and (6.11) imply that the DC tunneling current is given by  $I_t \propto V_t^{2g-1}$  that is *non-linear*. The non-linear  $I_t$ - $V_t$  curve is a consequence of the strong correlation in the FQH states.

The finite-temperature propagators can be obtained from the zero-temperature ones through a conformal transformation.<sup>49</sup> We find that (6.13) becomes

$$\begin{aligned} G_+(t) &= a^{-2}(\pi T/\omega_0)^{2g} |\text{sh}[\pi Tt]|^{-2g} e^{i\pi g \text{sgn}(t)} \\ G_-(t) &= a^{-2}(\pi T/\omega_0)^{2g} |\text{sh}[\pi Tt]|^{-2g} e^{-i\pi g \text{sgn}(t)} \end{aligned} \quad (6.16)$$

at finite temperatures. (6.16) implies

$$X_{ret}(\omega) = a^{-2} \omega_0^{-2g} (2\pi T)^{2g-1} B\left(g - i\frac{\omega}{2\pi T}, g + i\frac{\omega}{2\pi T}\right) \frac{\sin \pi\left(g + i\frac{\omega}{2\pi T}\right)}{\cos \pi g} \quad (6.17)$$

where  $B$  is the Beta function. We see that the differential conductance  $(dI_t/dV_t)|_{V_t=0}$  is proportional to  $T^{2g-2}$

Now let us consider a situation where  $V_t$  has an AC component:

$$V_t(t) = V_0 + V_1 \sin(\Omega t) \quad (6.18)$$

In this case the time average of  $f(\omega, t)$ ,  $\bar{f}(\omega) = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt f(\omega, t)$ , has a form

$$\bar{f}(\omega) = \sum_{n=-\infty}^{\infty} a_n \delta(eV_0 + n\Omega - \omega) \quad (6.19)$$

It is easy to see that  $a_n = a_{-n}$  are real and  $a_n$  only depend on the ratio  $\xi = eV_1/\Omega$ . In fact we have

$$a_n(\xi) = \frac{1}{4\pi^2} \int_0^{2\pi} dt dt' e^{in(t'-t)} e^{i\xi(\cos t' - \cos t)} \quad (6.20)$$

The DC component of the tunneling current  $I_t$  is found to be

$$I_t^{(DC)} \equiv \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt I_t(t) \propto \sum_{n=-\infty}^{\infty} a_n(\xi) (eV_0 + n\Omega)^{2g-1} \quad (6.21)$$

The exponent  $g$  in the electron operators is always larger than 1. For the  $\nu = \frac{p}{pq+1}$  ( $q = \text{even}$ ) hierarchical FQH state,  $g = q + 1$  (*e.g.*,  $g = 3$  for  $\nu = 1/3, 2/5, 3/7, \dots$ ). For the  $\nu = 1/3, 2/7, \dots$  FQH states, there are two edge branches moving in the opposite directions. In this case  $g$  depend on the interaction between the two branches (see (2.43) and (2.39)). The DC  $I_t$ - $V_t$  curves described by (6.17) at finite temperatures are plotted in Fig. 10 with  $g = 3$ . The  $I_t^{(DC)}$ - $V_0$  curve in (6.21) for a few values of  $eV_1/\Omega$  is plotted in Fig. 11. Again we have chosen  $g = 3$ .

To illustrate how the tunneling experiments can probe the internal correlation in the FQH states, let us consider three different  $\nu = 1/2$  FQH states. The first  $\nu = 1/2$  FQH state is the electron pairing state. The tunneling at low energies is governed by the pair tunneling. The exponent  $g$  for the electron pair propagator is 8 (from the  $\nu = 1/8$  boson Laughlin state). The second  $\nu = 1/2$  FQH state is described by the matrix  $K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

which corresponds the following wave function of a two layer system:

$$\prod (z_i^1 - z_j^1)^3 \prod (z_i^2 - z_j^2)^3 \prod (z_i^1 - z_j^2)$$

where  $z^1$  and  $z^2$  are the electron coordinates in the two layers. The exponent  $g$  in the electron propagator for such a state is given by  $g = 3$ . The third  $\nu = 1/2$  FQH state is the non-abelian state described by  $\chi_1\chi_2^2$  that has  $g = 3$ . Therefore the experimental measurement of  $g$  allow us to determine which  $\nu = 1/2$  state is realized in the sample.

We would like to remark that for generic FQH states, there may be many different electron operators on the edge. Different operators have different exponents in their propagators. It is the operator with smallest exponent dominate the tunneling at low energies. Therefore  $g$  is equal to the smallest exponent in the different electron propagators. The different singularities described by the different electron operators are in some sense similar to the different singularities in the electron propagator at  $k_F, 3k_F, etc.$  in the interacting 1D electron systems.

Now let us consider a more interesting device with a geometry as presented in Fig. 12. The low temperature transport of such a Hall bar is still governed by the tunneling between the two edges. For such a system the tunneling can be accomplished by moving one electron from one edge to the other, it can also be accomplished by moving one quasiparticle between the edges. This is because the edges are connected by the FQH state instead of vacuum. We know the quasiparticles in the FQH state correspond to vortices of unit flux. The quasiparticle tunneling across the FQH sample in some sense resembles the vortex tunneling across a superconducting stripe. The contribution from the electron tunneling is discussed above. In the following we will concentrate on the quasiparticle tunneling.

The quasiparticle tunneling operator is given by  $A = \Psi_{qR}^\dagger \psi_{qL}$  where  $\Psi_{q;R,L}$  are the quasiparticle operators on the edge R and L. The correlation of the tunneling operator  $A$  still has the form (6.13) but now  $g$  is the exponent in the quasiparticle propagator and in general is less than 1. We will assume the quasiparticle  $\Psi_{q;R,L}$  considered here has the minimum value of  $g$ , so that it dominates the tunneling.

The tunneling formula (6.7) also applies to the quasiparticle tunneling after replacing  $e$  by the quasiparticle charge  $e^*$ . Therefore our previous results for the electron tunneling remain to be valid for the quasiparticle tunneling once  $e$  is replaced by  $e^*$  and  $g$  is replaced by the exponent in the quasiparticle propagator.

The DC  $I_t$ - $V_t$  curves described by (6.17) at finite temperatures are plotted in Fig. 13 with  $g = 1/3$  (which corresponds to the  $\nu = 1/3$  Laughlin state). The  $I_t^{(DC)}$ - $V_0$  curve in (6.21) for a few values of  $eV_1/\Omega$  is plotted in Fig. 14. for the same value of  $g$ .

When  $g$  is less than  $\frac{1}{2}$ , the quasiparticle tunneling has very different behaviors than that of the electron tunneling. For example, the zero temperature DC  $I_t$ - $V_t$  curve is given by

$$I_t \propto |V_t|^{2g-1} \text{sgn}(V_t) \quad (6.22)$$

that *diverges* as  $V_t \rightarrow 0$ . Such a diverging tunneling curve resembles the tunneling curve between one dimensional superconductors (with algebraic decaying superconducting order parameters). The tunneling curve can be directly measured in experiments by measuring  $I = \sigma_{xy}V_t$  and  $V = I_t/\sigma_{xy}$  in Fig. 12. Here  $\sigma_{xy} = \nu \frac{e^2}{h}$  is the Hall conductance of the FQH state. We would like to remark that our discussions are base on the weak tunneling theory. Our results are valid only when the tunneling current is small:  $I_t \ll I$ , or equivalently,

$$I_t/V_t \ll \sigma_{xy}. \quad (6.23)$$

(6.22) violates (6.23) at small  $V_t$ . In this case (6.22) is no longer valid and the divergence is expected to be rounded off.

In presence of the AC component, we see that there are many resonance structures at integer values of  $e^*V_0/\Omega$ . Those structures provide a *direct* measurement of the quasiparticle charge  $e^*$ . The resonance structures in the tunneling between the two edge states resemble the Josephson effects between superconductors. The resonance structures in Fig. 14 reflect the existence of narrow band noise in DC transport, which is very similar to the narrow band noise in CDW transport. Probably a more direct way to measure the quasiparticle charge is to measure the frequency of the narrow band noise that is given by  $\Omega_{ns} = e^*V_t/\hbar$ .

The exponent  $g$  in the quasiparticle propagators is in general less than 1. For the  $\nu = \frac{p}{pq+1}$  ( $q = \text{even}$ ) hierarchical FQH state, the charge  $e^* = \nu$  quasiparticles have the minimum exponent  $g = \nu$  (e.g.,  $g = 1/3$  and  $e^* = 1/3$  for the  $\nu = 1/3$  FQH state. See the discussion at the end of the chapter 3). For the  $\nu = 2/3, 2/7, \dots$  FQH states,  $g$  again depend on the interaction between the two branches moving in the opposite directions.

The quasiparticles with larger exponents certainly also contribute to the tunneling, in particular at finite voltages and temperatures. Experimentally we can use the AC measurement to measure the quasiparticle charges, from which we can get some idea which quasiparticles contribute to the tunneling. Some times the second smallest exponent is quite close to the smallest exponent and it is difficult to say which one dominates the tunneling.

For the three  $\nu = 1/2$  FQH states discuss above, The quasiparticle with the dominate contribution to the tunneling has the following values of  $g$  and  $e^*$ . The pairing state has  $g = 1/8$  and  $e^* = 1/4$  due to its equivalence to the  $\nu = \frac{1}{8}$  Laughlin state of charge  $2e$  bosons. For the  $K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$  state, the exponent  $g$  is equal to  $\lambda_l$  given in (3.16) since the two edge branches move in the same direction. The minimum exponent  $g = 3/8$  is given by the charge  $e^* = 1/4$  quasiparticle. For the non-abelian state, the minimum  $g$  is given by (5.21) with  $l = 0$  and the charge by (5.14) and (5.18). Thus we have  $g = 1/2$  and  $e^* = 1/4$ . From (5.24) we see that the charge  $e^* = 1/2$  abelian quasiparticle also has the exponent  $g = 1/2$ .

In presence of the interedge interaction, the situation is more complicated. In general  $g$  depend on the strength of the interedge interaction when the two edges are parallel. The two terminal conductance of a quantum Hall sample with parallel edges will no longer be  $\nu \frac{e^2}{h}$  if there are interedge interactions.<sup>23</sup> The two terminal conductance  $\sigma$  and the exponent  $g$  for the charge  $\nu e$  quasiparticle (which exists in all the QH states) are related by

$$\sigma = g \frac{e^2}{h}$$

To minimize the effect of the interedge interaction, one can choose the device geometry as described in Fig. 12 and add metallic gate near the 2D electron gas to screen the Coulomb interaction.

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## FIGURE CAPTIONS

- Fig. 1: A QH state can be viewed as a droplet of an incompressible fluid. The gapless edge excitations are surface waves on the droplet.
- Fig. 2: A special edge configuration of the  $\nu = 2/5$  FQH states in which the electron droplet and the quasiparticle droplet are well separated.. The two edges are separated by the  $\nu = 1/3$  Laughlin fluid.
- Fig. 3 The spectrums of the edge excitations of the  $\nu = \frac{2}{5}$  FQH state obtained from (a) the effective theory and (b) the numerical calculation.
- Fig. 4 The spectrums of the edge excitations of the  $\nu = \frac{2}{3}$  FQH state obtained from (a) the effective theory and (b) the numerical calculation.
- Fig. 5 The energy spectrum of a QH system with 6 electrons in the first 22 orbits in the first Landau level (*i.e.*, the orbits with angular momenta  $0, 1, \dots, 21$ ). The interaction between electrons are described by the  $V_1$  pseudo potential.  $M$  is the total angular momentum of the 6 electrons. The first 100 energy levels are plotted for each  $M$ . The zero-energy states at  $M = 45, \dots, 51$  are degenerate with degeneracy  $1, 1, 2, 3, 5, 7, 11$ .
- Fig. 6 The equal time correlation of the electrons along the edge of the  $\nu = 2$  Laughlin state (for bosons) obtained from a Monte Carlo calculation. The Laughlin state contain 37 electrons. The solid line is the theoretical prediction (4.16).
- Fig. 7 The branching point of the edge states in QH samples can be classified into 8 classes. Each line represents all the edge branches moving in the indicated direction.
- Fig. 8 Quantum Hall samples containing two different filling fractions  $\nu$  and  $\nu_g$ .
- Fig. 9 Tunneling junction between two QH fluids.
- Fig. 10 The DC tunneling  $I_t$ - $V_t$  curve for the tunneling between two  $\nu = 1/3$  Laughlin states in Fig. 9 at finite temperatures,  $T = 0, 0.3, 0.6, 0.9, 1.2$ . In this case  $g = 3$ . We have chosen  $k_B = 1$ .  $eV_t$  and  $T$  are measured with same energy unit.
- Fig. 11 The  $I_t^{DC}$ - $V_0$  curve for the tunneling between two  $\nu = 1/3$  Laughlin states in Fig. 9. The voltage  $V_t$  has an AC component of amplitudes  $eV_1/\Omega = 0, 2, 4, 6$ .
- Fig. 12 A Hall bar with a narrow neck. The electrons in the shaded region form a QH state.
- Fig. 13 The DC  $I_t$ - $V_t$  tunneling curve for the tunneling between the two edges of the same  $\nu = 1/3$  Laughlin state in Fig. 12 at finite temperatures,  $T = 0, 0.3, 0.6, \dots, 1.8$ . In this case  $g = 1/3$  and  $e^* = \nu e$  is the quasiparticle charge. We have chosen  $k_B = 1$ .  $e^*V_t$  and  $T$  are measured with same energy unit.
- Fig. 14 The  $I_t^{DC}$ - $V_0$  curve for the tunneling between the two edges of the same  $\nu = 1/3$  Laughlin state in Fig. 12 with different AC components in  $V_t$ . We have chosen the temperature  $T$  to be zero and  $\hbar = 1$ .