Edge excitations in the fractional quantum Hall states at general filling fractions*

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ABSTRACT: We propose a scheme to construct the edge states for the Fractional Quantum Hall (FQH) states obtained by the Jain's scheme. The low energy effective theory of the edge excitations is obtained. We calculate the number of the branches and the electron propagators of the edge states for various FQH states. We demonstrate that two FQH states with the same filling fraction may have different topological orders and may support different edge excitations.

PACS numbers: 73.20.Dx, 73.20.Mf, 73.40.Kp

* Published in *Mod. Phys. Lett. B* **5** 39 (1991)

Recently the edge excitations in the Fractional Quantum Hall (FQH) states have attracted a lot of attentions. The dynamical properties of the edge excitations in FQH states have been studied in Ref. 1,2,3,4,5 with help of the Kac-Moody (K-M) algebra. It is shown that the electrons on the edge of a FQH states form a new kind of quantum liquid state[4] which cannot be described by the Fermi liquid theory. DC transport properties of the edge channels were discussed in Ref. 6,7,8. The experimental and the theoretical results about the edge excitations in the Integral Quantum Hall (IQH) states can be found in Ref. 9 and Ref. 10.

In Ref. 4 it is argued that the QH states with filling fraction $\nu \neq 1/m$ [11,12] must have more than one branch of the edge excitations. In this letter we will discuss in detail the edge excitations in these generic QH states. Some aspects of the edge excitations of the hierarchy QH states have been studied in Ref. 8. Here we will give a different approach which leads to a more detailed picture about the edge excitations.

The properties of the edge excitations are not only determined by the filling fraction, but also determined by the internal structures of the FQH wave function. Such internal structures may be called the topological orders in the FQH states[13,14]. The filling fraction alone does not provide enough information for the edge excitations. To determine the properties of the edge excitations, we must also specify the topological orders in the FQH states. In this letter we will specify the topological orders by specifying a particular way to construct the FQH wave functions. We are going to use the Jain's sheme proposed in Ref. 12. More general characterizations of the topological orders can be found in Ref. 13,14.

Let us first review Jain's construction using a FQH state with filling fraction $\nu = \frac{2}{5}$ as an example. His trick is to view the electrons as bound states of a charge $\frac{e}{5}$ fermion and a charge $\frac{4e}{5}$ boson. The filling fraction ν and the electron density n_e is related by $\nu = \frac{1}{e}\frac{hc}{B}n_e$. Since the $\frac{e}{5}$ fermions and the $\frac{4e}{5}$ have the same densities as the electrons $n_f = n_b = n_e$, the $\frac{e}{5}$ fermions have a filling fraction $\nu_f = \frac{5}{e}\frac{hc}{B}n_f = 2$ and the $\frac{4e}{5}$ bosons have a filling fraction $\nu_b = \frac{5}{4e}\frac{hc}{B}n_b = \frac{1}{2}$.

We start from the limit in which the fermions and bosons have no interactions and there is a hardcore repulsive interaction between the bosons. In this limit the bosons form a $\nu_b = \frac{1}{2}$ FQH state described by the Laughlin's wave function

$$\Psi_{1/2}(z_i^b) = [\prod_{i<j}(z_i^b - z_j^b)^2]exp(-\frac{1}{4}\sum_i \frac{4e}{5}\frac{B}{\hbar c}|z_i^b|^2) \tag{1}$$

and the fermions form a $\nu_f = 2$ IQH state described by wave function $\Psi_2(z_i^f)$. Here z_i^b and z_i^f are coordinates of the bosons and the fermions. Now let us bind the fermion and the boson together. This is equivalent to suppress the $n_b - n_f$ fluctuations (*i.e.*, $n_b - n_f = 0$). In this case the low lying excitations are described by the wave functions of the bound states, or equivalently, the wave function of the electrons. The above picture leads to a trial wave function for the ground state

$$\Psi(z_i) = \Psi_{1/2}(z_i^b)\Psi_2(z_i^f)|_{z_i^b = z_i^f = z_i} \tag{2}$$

where z_i are coordinates of the electrons and $z_i^b = z_i^f$ is the projection to the $n_f - n_b = 0$ sector. (2) is exactly the wave function constructed by Jain for the $\nu = \frac{2}{5}$ FQH state.

The quasi particle excitations with electric charge $\frac{m}{5}e$ can be shown to have a statistics $\theta = \frac{3}{5}\pi m^2$ in such a state.[15] In our notation Ψ_ν is the QH wave function with a filling fraction ν for the fictitious particles. The charge of of the fictitious particles are chosen such that $\Psi_{1/2}$ and Ψ_2 have the same particle density despite they have different filling fractions.

The essence of the above construction is the following. We first decompose the electrons into fictitious particles. Those fictitious particles have simple filling fractions such that one can easily write down their wave functions. By introducing the fictitious particles, we introduce some unphysical degrees of freedom. We need to make a projection to project away all the unphysical degrees of freedom and to obtain the correct physics for the electrons. In particular the ground state wave function is obtained by doing such a projection (see (2)). The Jain's scheme[12] is very convenient for the construction of the edge excitations. This is because the projection can be done at the effective theory level.

Let us first discuss the edge excitations in the $\nu = \frac{n}{mn+1}$ FQH state (m is an even integer). The electrons are decomposed into fermions with a charge $q_1 = \frac{\nu}{n}e$ and bosons with a charge $q_2 = \nu m e$. The electron wave function of the FQH state is given by

$$\Psi(z_i) = \Psi_n(z_i)\Psi_{1/m}(z_i) \tag{3}$$

Let us assume that the FQH state has a disk-like geometry. The boundary of the disk is parameterized by x. In Ref. 1,4 It is shown that the edge excitations are described by the $U(1)$ K-M algebras. Those edge excitations can be regarded as surface waves propagating along the boundary of the incompressible QH fluid.[2,5] The surface wave can be described by the "edge density" $\rho(x) = n_e h(x)$ where $h(x)$ is the displacement of the edge. Note ρ is an one dimensional density with dimension 1/[length].

Before the projection, the charge q_2 bosons form a $\nu_b = \frac{1}{m}$ FQH state and support a single branch of edge excitations. Those excitations are described by the the following K-M algebra[1,3,4]

$$[\rho_{0,k}, \rho_{0,k'}] = \frac{1}{m}\frac{k}{2\pi}\delta_{k+k'} \tag{4}$$

where ρ_0 is the edge density of the bosons. The boson creation operator on the edge is given by[4] $\psi_0 =: e^{im\phi_0}:$ where $\partial_x\phi_0 = 2\pi\rho_0$. ψ_0 carries an electric charge q_2. The charge q_1 fermions form a $\nu_f = n$ IQH states which support n branches of edge excitations.[10] Those edge excitations are described by

$$[\rho_{i,k}, \rho_{i',k'}] = \frac{k}{2\pi}\delta_{k+k'}\delta_{i,i'}, \quad i,i' = 1,2,\ldots,n \tag{5}$$

The fermion creation operators on the edge are given by $\psi_i =: e^{i\phi_i}:$ with $\partial_x\phi_i = 2\pi\rho_i, \ \ i = 1,\ldots,n$. They carry an electric charge q_1. ρ_i and ψ_i are the edge density and the fermions of the ith Landau level. The coupling between the edge densities and the external electric potential is given by

$$(q_2\rho_0 + \sum_{i=1}^{n} q_1\rho_i)A_0 \tag{6}$$

Before the projection, the Hilbert space of the edge excitations is generated by ρ_i and ψ_i with $i = 0,1,\ldots,n$, which contains $n+1$ branches.

Because the fluctuations associated with $\tilde{\rho} = \rho_0 - \sum_{i=1}^n \rho_i$ $(= \rho_b - \rho_f)$ are unphysical, we should remove all such fluctuations in order to obtain the correct edge excitations. To accomplish this, we will first specify the physical operators. A physical operator must not create any fluctuations associated with $\tilde{\rho}$. Hence a physical operator must commute with $\tilde{\rho}$:

$$[\hat{O}_{phy}, \tilde{\rho}] = 0 \tag{7}$$

One can easily check that the following edge density operators are physical

$$\begin{aligned} j_0 =& \sqrt{\nu}(m\rho_0 + \frac{1}{n}\sum_{i=1}^n \rho_i) \\ j_i =& \sum_{j=1}^n a_i^j \rho_j, \quad i = 1, \dots, n-1 \end{aligned} \tag{8}$$

where $\sum_{j=1}^n a_i^j = 0$ and $\sum_{j=1}^n a_i^j a_{i'}^j = \delta_{i,i'}$. Similarly the charged physical operators (with the minimum charge) are given by $\Psi_i =: e^{i(m\phi_0 + \phi_i)}:$, $i = 1, \dots, n$ The operators Ψ_i carry an electric charge e. They are just the electron creation operators on the edge. The Hilbert space of the physical edge excitations is generated by j_i and Ψ_i, thus contains n branches. We find that the edge excitations of the $\nu = \frac{n}{mn+1}$ FQH state have n branches. We would like to remark that there is a gauge symmetry in the above construction. The gauge symmetry is generated by $\tilde{\rho}$. (7) is just the gauge invariant condition of the physical operators. The appearance of the gauge symmetry is due to the introduction of the unphysical degrees of freedom.[16]

The above result can be understood from the microscopic wave function (3). We may view $\Psi_{1/m}$ as an operator which maps the wave function Ψ_n of the $\nu = n$ IQH state to the wave function of the $\nu = \frac{n}{mn+1}$ FQH state.[17] This operator also maps the n branches of edge excitations in the IQH state to the n branches of edge excitations in the FQH state. Using this correspondence we can easily write down the microscopic wave functions for the edge excitations in the FQH state. A discussion about the similarity between the $\nu = n$ IQH state and the $\nu = \frac{n}{mn+1}$ FQH state can be found in Ref. 17.

From (8), (4) and (5) we see that the physical edge density operators satisfy the following K-M algebra $[j_{i,k}, j_{i',k'}] = \frac{k}{2\pi}\delta_{k+k'}\delta_{i,i'}$ From (6) we find that only j_0 couples to the electric potential $e\sqrt{\nu} j_0 A_0$. Using the algebra (4) and (5) we can easily calculate the equal time correlations between Ψ_i and $\Psi_i^\dagger$: $\langle \Psi_i^\dagger(x)\Psi_j(y)\rangle \propto (x-y)^{-m-1}\delta_{i,j}$. The electronic state on the edge is definitely not a Fermi liquid due to the anomalous exponent in the correlation functions. We can also show that $\{\Psi_i(x), \Psi_j(y)\} = 0$. Therefore Ψ_i are indeed fermionic operators.

In the following we will discuss the dynamical properties of the edge excitations. For simplicity we will only discuss the $\nu = \frac{2}{5}$ FQH state. In this case the edge excitations contain two branches generated by j_0 and $j_1 = \sqrt{\frac{1}{2}}(\rho_1 - \rho_2)$. The two electron operators are given by $\Psi_{1,2} =: e^{i(\sqrt{\frac{5}{2}}\Phi_0 \pm \sqrt{\frac{1}{2}}\Phi_1)}:$ where $\partial_x \Phi_i = 2\pi j_i$, $\quad i = 0, 1$. The most general low energy effective Hamiltonian for the edge excitations has a form

$$H = \frac{1}{\pi}(v_0 j_0^2 + v_1 j_1^2 + V j_0 j_1) \tag{9}$$

The higher order terms in j_i are irrelevant operators at low energies. When $V = 0$ it can be shown that $j_{i,k}$, $i = 0, 1$ create an energy eigenstate with energy $\epsilon = v_i k$. ($j_{i,k}|_{k<0}$ annihilate the ground state.) Therefore j_0 and j_1 generates two branches of the edge excitations with velocities v_0 and v_1 respectively. In this case only the first branch couples to the external electromagnetic field and the edge magnetoplasmon experiments can only observe a single resonance peak. However when $V \neq 0$ the energy eigenstates are always created by a mixture of j_0 and j_1. In this case both branches couple to the electromagnetic field and the edge magnetoplasmon experiments should see two resonance peaks. We would like to mark that in (9) we have ignored the long range Coulomb interaction between charge fluctuations. The Coulomb interaction can be screened by a metal plate near the FQH sample. If we want to include the Coulomb interaction, a term $\int dx\, dy \nu e^2 \frac{j_0(x) j_0(y)}{|x-y|}$ should be added to the Hamiltonian (9). In this case the dispersion relation becomes $\epsilon \propto -k \ln(k\xi)$.

The electron propagators are given by

$$G_i(t,x) \propto (\frac{1}{x - \tilde{v}_0 t})^{\frac{5}{2}+\alpha_i} (\frac{1}{x - \tilde{v}_1 t})^{\frac{1}{2}-\alpha_i}, \quad i = 1,2 \tag{10}$$

where $\alpha_1 = \sqrt{5}\cos\theta\sin\theta - 2\sin^2\theta$ and $\alpha_2 = \sqrt{5}\cos\theta\sin\theta + 2\sin^2\theta$ with θ satisfying $\text{th}2\theta = \frac{V}{v_0 - v_1}$. $\tilde{v}_0$ and $\tilde{v}_1$ are the velocities of the two branches when $V \neq 0$. (10) implies that the electron operators have a dimension $\frac{3}{2}$ under the renormalization group scaling $(x,t) \to (\eta x, \eta t)$. The electron spectral function $n(\omega, k)$ obtained from (10) satisfies

$$N(\omega) = \int dk n(\omega, k) \propto (\mu - \omega)^2 \theta(\mu - \omega) \tag{11}$$

where μ is the chemical potential. $N(\omega)$ can be measured by tunneling experiments. For a metal-insulator-FQH junction we find that $dI/dV \propto V^2$ (For the $\nu = \frac{n}{mn+1}$ FQH state we have $dI/dV \propto V^m$). For FQH-insulator-FQH junction the differential conductance is $dI/dV \propto V^4$.

The above results can also be obtained from a field theory calculation. Spliting electrons into the charge $q_1 = \frac{\nu}{n}$ fermions ψ and charge $q_2 = \nu m$ bosons φ can be realized through the Lagrangian

$$\begin{aligned} \mathcal{L} =& i\varphi^\dagger(\partial_0 - iq_2 A_0 - ia_0)\varphi - \frac{1}{2m}\varphi^\dagger(\partial_i - iq_2 A_i - ia_i)^2\varphi \\ &+ i\psi^\dagger(\partial_0 - iq_1 A_0 + ia_0)\psi - \frac{1}{2m'}\psi^\dagger(\partial_i - iq_1 A_i + ia_i)^2\psi \end{aligned} \tag{12}$$

a_μ is the Lagrangian multiplier which set the boson current and density equal the fermion current and density. Thus integrating out a_μ binds the bosons and the fermions back into the original electrons and corresponds to the projection to the $n_f = n_b$ sector. Now let us first integrate out the boson field ϕ and the feimion field ψ. This lead to the following effective action

$$\begin{aligned} S_{\text{eff}} =& \int dt d^2x \left[\frac{n}{4\pi}(q_1 A_\mu - a_\mu)\partial_\nu(q_1 A_\lambda - a_\lambda)\epsilon^{\mu\nu\lambda} \frac{1}{4\pi m}(q_2 A_\mu + a_\mu)\partial_\nu(q_2 A_\lambda + a_\lambda)\epsilon^{\mu\nu\lambda} \right] \\ &+ \int_{\text{edge}} dt dx [\mathcal{L}^f_{\text{edge}} + \mathcal{L}^b_{\text{edge}}] \end{aligned} \tag{13}$$

since the fermion form a $\nu = n$ IQH state and the boson form a $\nu = 1/m$ FQH state. In (13)

$$\mathcal{L}^f_{\text{edge}} = \sum_{a=1}^{n} i\psi^\dagger_a(\partial_0 - iq_1 A_0 + ia_0 - v_a\partial_x + iq_1 v_a A_x - iv_a a_x)\psi_a \tag{14}$$

describes the n branches of the edge excitations of the fermion IQH state, and $\mathcal{L}^b_{\text{edge}}$ is a chiral boson Lagrangian discussed in Ref. 4 which describe the edge excitations of the boson FQH state. If a_μ was set to zero, (14) would describe $n+1$ branches of edge exitations. But for dynamical a_μ field (14) is just a generalization of the chiral Schwinger model studied in Ref. 18. It was shown that one branch of the edge excitation coupled to a_μ is eaten by the gauge field and obtains a finite energy gap (the chiral version of the Higges mechanism). Thus the branch associated with density fluctuations $\tilde{\rho} = \rho_0 - \sum_{i=1}^{n} \rho_i$ disappears from the low energy spectrum. Only n branches of edge excitations with $\tilde{\rho} = 0$ are left at low energies. This is excatly the results that we obtained before.

The above results can be easily generalized to the FQH state described by the following wave function[12]

$$\Psi(z_i) = \Psi_{1/l}(z_i) \prod_{i=1}^{p} \Psi_{n_i}(z_i) \tag{15}$$

where l is even (odd) if p is odd (even). The filling fraction is $\nu = (l + \sum_{i=1}^{p} \frac{1}{n_i})^{-1}$. The edge excitations have $1 + \sum_i (n_i - 1)$ branches. The electron operators on the edge have a dimension $\frac{l+p}{2}$ or

$$\langle \Psi^\dagger(x)\Psi(y)\rangle \propto (\frac{1}{x-y})^{l+p} \tag{16}$$

The averaged spectral function has a form

$$N(\omega) \propto (\mu - \omega)^{l+p-1}\theta(\mu - \omega). \tag{17}$$

Notice that when $n_i = 1$ or when $p = 0$, (15) becomes the Laughlin wave functions. In this case the above results reduce to the results obtained in Ref. 4. It is clear that the construction also applies to even more general hierarchy FQH states. For example Ψ_{n_i} in (15) does not have to be an IQH wave function. It can be a FQH wave function with a filling fraction $\nu = 1 \pm \frac{1}{l}$. The edge excitations for such FQH states have been studied in Ref. 3.

A $\nu = \frac{2}{7}$ FQH state can be obtained by choosing $l = 2,\ p = 3,\ n_i = 2$. For such a FQH state, there are four branches of edge excitations. Another $\nu = \frac{2}{7}$ FQH state can also be obtained by choosing $l = 3,\ p = 2,\ n_i = 4$. In this case the edge excitations have seven branches. This result suggests that the two FQH states have different topological orders despite they have the same filling fraction. The different topological orders are reflected in the different edge excitations. The third way to construct a $\nu = \frac{2}{7}$ FQH state is to choose $l = 2,\ p = 1, n = \frac{2}{3}$. In this case the edge excitations contain only two branches if we use the results in Ref. 3,8 for the $\nu = \frac{2}{3}$ FQH state.

In this paper we propose a method to construct the edge excitations for generic FQH states. We calculate the number of the branches and the electron propagators of the edge states for various FQH states. The number of the branches and the dimension of the electron operators (or the exponents in the electron correlation function (16) and in the spectral function (17)) are universal properties of the FQH states.[19] They are invariant under arbitrary perturbations and are independent of the details of the electron interactions and the edge potentials. We may use those numbers to characterize the topological orders in the FQH states. We also demonstrate that the FQH states with a given filling fraction may have different topological orders. We show that different topological orders result in different edge excitations. Therefore the edge excitations provided a practical way to measure the topological orders in the FQH states.

Our results suggest that the incompressible QH states can be regarded as being made of many components of incompressible fluids. Each component give rise to one branch of the edge excitations. In the IQH states each filled Landau level correspond to one component of incompressible fluid. Here we show that the FQH states may contain many components even when $\nu < 1$.

In Ref. 3 the $\nu = 1 - \frac{1}{m}$ FQH states are shown to have two branches of edge excitations with opposite velocities. The electron propagator is shown to have a form

$$G(x,t) = \frac{1}{(x - v_R t)^{2h} (x + v_L t)^{2\bar{h}}} \tag{18}$$

The exponent h and $\bar{h}$ by themselves are not universal and depend on the electron interactions. However, the difference $h - \bar{h}$ is universal.[20] For the $\nu = 1 - \frac{1}{m}$ FQH states, there are two different electron operators on the edge with $2h - 2\bar{h}$ given by $-m$ and 1.[20] MacDonald has constructed the edge states of the hierarchy FQH states by using the particle-hole duality. But his characterization of the edge excitations is very different from ours. It is not clear how to compare his results with the results obtained here. However for the $\nu = \frac{n}{2n+1}$ FQH states we obtain the same number of the branches as MacDonald does. A discussion about the edge excitations in the hierarchy FQH states constructed in Ref. 11 will appear elsewhere.[21]

This work was supported by grant DE-FG02-90ER40542.

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