Quantum Field Theory of Many-body Systems
– from the Origin of Sound
to an Origin of Light and Fermions

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Chapter 11

Tensor category theory of string-net condensation

Extended objects, such as strings and membranes, have been studied for many years in the context of statistical physics. In these systems, quantum effects are typically negligible, and the extended objects can be treated classically. Yet it is natural to wonder how strings and membranes behave in the quantum regime. In this chapter, we will investigate the properties of one dimensional, string-like, objects with large quantum fluctuations. Our motivation is both intellectual curiosity and (as we will see) the connection between quantum strings and topological/quantum orders in condensed matter systems.

It is useful to organize our discussion using the analogy to the well understood theory of quantum particles. One of the most remarkable phenomena in quantum many-particle systems is particle condensation. We can think of particle condensed states as special ground states where all the particles are described by the same quantum wave function. In some sense, all the symmetry breaking phases examples of particle condensation: we can view the order parameter that characterizes a symmetry breaking phase as the condensed wave function of certain "effective particles." According to this point of view, Landau's theory [Landau (1937)] for symmetry breaking phases is really a theory of "particle" condensation.

The theory of particle condensation is based on the physical concepts of long range order, symmetry breaking, and order parameters, and the mathematical theory of groups. These tools allow us to solve two important problems in the study of quantum many-particle systems. First, they lead to a classification of all symmetry-breaking/particle-condensed states. For example, we know that there are only 230 different crystal phases in three dimensions. Second, they provide insight into the quasiparticle excitation spectrum. The collective excitations above the ground state are described by fluctuations of the amplitude of the condensed "particles" (i.e. the fluctuations of the order parameter). In many cases, symmetry breaking allows us to derive the quantum numbers of these collective excitations (or quasiparticles) and predict whether they are gapped or gapless.

Given the importance of the concept of particle condensation, it is natural to consider the analogous concept of "string condensation." What do we mean by "string condensation"? A natural definition is that a string condensed state is a ground state that (a) is formed by many large strings, whose sizes are of order of the size of the system, and (b) is a superposition of many different large string configurations. In other words, a string condensed state is a quantum liquid of large strings.

We would like to have a theory of string condensation which is as powerful as the analogous theory of particle condensation. That is, we would like to have a general framework for (1) characterizing and classifying different string condensed states, and (2) determining the physical properties of the collective excitations of string condensed states.

Some progress has been made towards these goals. Much of this progress has occurred in three areas of research: (1) the study of topological phases in condensed matter systems such as FQH systems [Wen and Niu (1990); Blok and Wen (1990); Read (1990); Fröhlich and Kerler (1991)], quantum dimer models [Rokhsar and Kivelson (1988); Read and Chakraborty (1989); Moessner and Sondhi (2001); Ardonne *et al.* (2004)], quantum spin models [Kalmeyer and Laughlin (1987); Wen *et al.* (1989); Wen (1990); Read and Sachdev (1991); Wen (1991*a*); Senthil and Fisher (2000); Wen (2002*b*); Sachdev and Park (2002); Balents *et al.* (2002)], or even superconducting states [Wen (1991*b*); Hansson *et al.* (2004)], (2) the study of lattice gauge theory [Wegner (1971); Banks *et al.* (1977); Kogut and Susskind (1975); Kogut (1979)], and (3) the study of quantum computing by anyons [Kitaev (2003); Ioffe *et al.* (2002); Freedman *et al.* (2002)]. The phenomenon of string condensation is important in all of these fields, though the string picture is often de-emphasized.

Some of the early work in this area was in the study of topological order - a kind of order that can occur in exotic condensed matter systems [Wen (1995)]. Ref. [Wen (1990)] used ground state degeneracy, particle statistics, and edge excitations to partially characterize topologically ordered states. Later, Ref. [Wen (2002*b*, 2003*b*)] attempted to characterize and classify quantum order – a generalization of topological order to gapless phases – using the projective symmetry group (PSG) formalism. String condensed states are typically topologically ordered. So these results can be viewed as partial classifications of string condensations.

The collective fluctuations of string condensation have also been analyzed to some extent. Just as in particle condensation, these fluctuations give rise to new emergent quasiparticle excitations. However, the similarity ends here. The emergent quasiparticles in particle condensed states are always scalar bosons. In contrast, the emergent particles in string condensed states are (deconfined) gauge bosons[Banks *et al.* (1977); Foerster *et al.* (1980); Wen (2002*a*); Motrunich and Senthil (2002); Wen (2003*a*)] and fermions[Levin and Wen (2003); Wen (2003*b*)]. Fermions can emerge as collective excitations of purely bosonic models! (The emergence of deconfined fermions/anyons from purely bosonic models was first studied in 2+1 dimensional models[Arovas *et al.* (1984); Kalmeyer and Laughlin (1987); Wen *et al.* (1989); Read and Sachdev (1991); Wen (1991*a*); Moessner and Sondhi (2001); Kitaev (2003)]. In 2+1 dimension, one can understand the emergent fermions using a flux binding picture. However, beyond 2+1 dimension, one needs to use the string picture to understand the emergence of fermions. In fact, the string picture works in any dimension.) As in the case of particle condensation, the PSG that characterizes different string condensed states can also protect the gaplessness of the emergent gauge bosons and fermions [Wen (2002*b*); Wen and Zee (2002)].

Lattice gauge theory has provided additional insights into string condensation. It is well known that Abelian gauge theory has a dual description in terms of closed strings - each closed string corresponds to an electric flux line [Wegner (1971); Banks *et al.* (1977); Kogut (1979)]. Ref. [Kogut and Susskind (1975)] showed that non-Abelian lattice gauge theory also has a dual description. This description involves more general 1-dimensional objects: strings with branching. We will refer to these networks of strings as "string-nets". Ref. [Kogut and Susskind (1975)] showed that the string-net condensed phase in the string model corresponded to the deconfined phase of the gauge theory, while the normal phase in the string model corresponded to the confining phase of the gauge theory. This suggests that string-nets are perhaps a more natural object to study then closed strings.

These results demonstrate that string (or string-net) condensation is associated with a host of interesting physical phenomena – from anyons and fractionalization to emerging gapless gauge bosons and fermions. However, they fail to provide a unified framework.

In this chapter, we will attempt to describe a unified theory for the simplest type of string-net condensed phase – topological string-net phases with no broken or unbroken symmetry. We will present a general theory of these topological string-net condensates that is analogous to the well known theory of particle condensation. We will show that, just as the low energy effective theories for particle condensation are Ginzburg-Landau theories [Ginzburg and Landau (1950)], the effective theories for topological string-net condensation are topological field theories [Witten (1989*b*)]. Just as long range order is the basic physical concept underlying particle condensation, topological order [Wen (1995)] is fundamental to topological string-net condensation. Furthermore, just as group theory is the mathematical framework behind particle condensation, something called "tensor category theory" is the framework underlying topological string-net condensation.

As in particle condensation, this framework will provide us with (1) a partial classification of the string condensates and (2) a method for determining the physical properties (e.g. the statistics) of the collective excitations.

Our approach, inspired by Ref. [Kogut and Susskind (1975); Witten (1989*a*, 1990); Freedman *et al.* (2003*b*,*a*)], is based on the string-net wave function. We construct "fixed-point" wave functions for a large class of string condensed phases. The "fixed-point" wave functions are special string wave functions with the property that they look the same at all length scales. We expect that if we could do an RG calculation for ground state wave functions, then all the states in the string condensed phase would flow to the "fixed-point" wave function at long distances. Thus, we believe that the wave functions capture the universal properties of the corresponding phases. Each "fixed point" wave function is associated with a solution to a complicated non-linear equation. Hence, there is a one-to-one correspondence between string condensates and solutions to this equation. (Solutions to this equation, in turn, correspond to "tensor categories" [Kassel (1995)])

In addition to a wave function, our construction also yields an exactly soluble lattice Hamiltonian (with the "fixed-point" wave function as its ground state) for each of the string condensed phases. Such an exactly soluble lattice Hamiltonian can be viewed as the fixed point Hamiltonian at the end of the RG flow. Using these Hamiltonians, we can find all the quasiparticle excitations and calculate their statistics. We find that the low energy effective theories for these states are topological field theories. Hence, the results obtained here can be viewed as a (partial) classification and analysis of topological field theories.

Our construction yields exactly soluble spin Hamiltonians for a large class of topological phases. These Hamiltonians are a direct generalization of the exactly soluble lattice gauge theory Hamiltonians discussed in [Wegner (1971); Kitaev (2003)]. In addition to gauge theories, our models describe many other topological field theories, including all doubled Chern-Simons theories (Abelian and non-Abelian), the gauge theories with emergent fermions.

11.1 Particle condensation

To describe the logic of our construction in simple setting, let us consider particle condensation first. A simple example of particle condensation is given by the transverse-field Ising model

$$H_{\text{Ising}} = h \sum_{\boldsymbol{i}} \sigma^z_{\boldsymbol{i}} - t \sum_{\boldsymbol{ij}} \sigma^x_{\boldsymbol{i}} \sigma^x_{\boldsymbol{j}}. \qquad (11.1.1)$$

The model has two $T = 0$ phases. When $h \gg t$, the model is in the state with all spins pointing down. As we decrease h/t, the model experiences a symmetry breaking transition. When $h/t = 0$, the model is the spin with all spins pointing in x-direction or $-x$-direction. The state with all spins pointing in x-direction is given by $|\Phi\rangle = \otimes_{\boldsymbol{i}}(|\uparrow\rangle_{\boldsymbol{i}} + |\downarrow\rangle_{\boldsymbol{i}})$. Note that it has the same amplitude for any σ^z-spin configurations.

We may view the spin-1/2 system as a hardcore boson system. $|\downarrow\rangle$ is viewed as an empty site and $|\uparrow\rangle$ an occupied site. In the boson picture, $2h$ represents the energy cost to create a boson and t is the boson hopping amplitude. When $h \gg t$, it costs a lot of energy to create a boson and the ground state of the system contains few fluctuating bosons (note that the boson number is not conserved.) When $t \gg h$, the hopping term dominates. The system prefer to create bosons and form superpositions between states connected by the hopping term to lower its energy. So the ground state is filled with bosons. When $h/t = 0$, the ground state $|\Phi\rangle$ can be described by a boson wave function $\Phi(\boldsymbol{i}_1, ..., \boldsymbol{i}_N)$ =constant, where $\boldsymbol{i}_n$ are coordinates of the bosons (or the up-spins). Since $\Phi\rangle$ has the same amplitude for any boson configurations in real space, it is a boson condensed state. Thus we can say that the ground state of H_{Ising} corresponds to a condensation of σ^z-spin if $t \gg h$. We note that the ideal boson condensed wave function $\Phi(\boldsymbol{i}_1, ..., \boldsymbol{i}_N)$ =constant is topological. The amplitude of a boson configuration $\{\boldsymbol{i}_1, ..., \boldsymbol{i}_N\}$ does not depend on the size and shape of the boson configuration.

We can use local rules to describe the above topological wave function for the σ^z-spin condensed state. The local rules specify the relation between the amplitudes of different spin configurations in the ground state. Let us use $\Phi(\alpha_{\boldsymbol{i}}\alpha_{\boldsymbol{j}}...)$ to describe the amplitude for a spin configuration with spin $\alpha_{\boldsymbol{i}}$ on site $\boldsymbol{i}$ and spin $\alpha_{\boldsymbol{j}}$ on site $\boldsymbol{j}$ where $\alpha =\uparrow$ or $\downarrow$. Here, we have used "..." to represent the spins on other sites. The local rules that describe the topological wave function are given by

$$\Phi(\uparrow\downarrow ...) = \Phi(\downarrow\uparrow ...), \qquad \Phi(\uparrow\uparrow ...) = \Phi(\downarrow\downarrow ...). \tag{11.1.2}$$

for any pairs of nearest neighbor sites $\boldsymbol{i}$ and $\boldsymbol{j}$. The above condition on the spin wave function can be represented by a projector. Define $P_{\boldsymbol{ij}}$ by

$$P_{\boldsymbol{ij}}|\Phi\rangle = |\Phi\rangle, \qquad P_{\boldsymbol{ij}} \equiv \frac{1}{2}(1 + \sigma^x_{\boldsymbol{i}}\sigma^x_{\boldsymbol{j}})$$

Then $P_{\boldsymbol{ij}}$ projects into the subspace of the wave functions that satisfy the local rules (11.1.2).

Here we would like to introduce two concepts: self-consistency and completeness of local rules. A set of local roles is self-consistent if there is at least one wave function that satisfies the local rules. A set of local rules is complete if there are only a finite number of linearly independent wave functions that satisfy the local rules. Because of the close relation between local rules and projectors, a set of self-consistent and complete of the local rules can be represented by a set of projectors $P_{\boldsymbol{ij}}$ that satisfy

$$[P_{\boldsymbol{ij}}, P_{\boldsymbol{kl}}] = 0, \qquad \mathrm{Tr} \prod_{\langle \boldsymbol{ij} \rangle} P_{\boldsymbol{ij}} = \text{finite}$$

The rules (11.1.2) are indeed self consistent and complete. This is because the state with all spins in the σ^x-direction satisfies the two local rules. On any connected lattice, there are only two states that satisfy the local rules. One state has an even number of up-spins and the other has an odd number of up-spins.

Using the projectors $P_{\boldsymbol{ij}}$, we can construct an exactly soluble Hamiltonian

$$\tilde{H}_{\mathrm{Ising}} = \sum_{\langle \boldsymbol{ij} \rangle} (1 - P_{\boldsymbol{ij}})$$

whose ground state has a condensation of σ^z-spin. Since the local rules are self-consistent, there is at least one state that has zero energy. Such states are the ground states. Since the local rules are complete, the ground state degeneracy is finite even when the lattice size approaches infinite. The constructed Hamiltonian $\tilde{H}_{\mathrm{Ising}}$ is essentially the Hamiltonian of the Ising model H_{Ising}.

To summarize, we can use a set of local rules to describe a σ^z-spin condensed state. Using the local rules and the associated projectors, we can construct a Hamiltonian whose ground states satisfy the local rules and have a σ^z-spin condensation, provided that the local rules are self consistent and complete. In the following, we will use a similar approach to study string-net condensation.

11.2 Picture of string-net condensation

To understand string-net condensation, let us consider a string-net model whose Hilbert space is made up of linear superpositions of "string-net configurations." String-net configurations are collections of curves in space which may contain ends and branches (see Fig. 11.1).[1] The curves represent strings, and typically they are labeled to indicate what type of string they are. For simplicity, we will focus on the case where at most three curves (or strings) are allowed to meet at a point.

[1]Since open strings are allowed, we will see later that, on a lattice, the string-net model is just an ordinary spin model, and any spin model can be regarded as a string-net model.

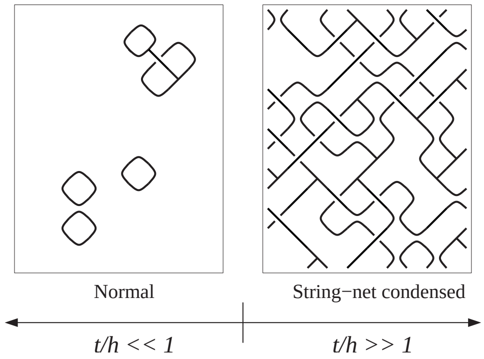


Figure 11.1: A schematic phase diagram for the generic string-net Hamiltonian (11.2.1). When t/h is small, the system is in the normal phase. The ground state is essentially a state with a few small string-nets. When t/h is large, the string-nets condense and large fluctuating string-nets fill all of space.

The string-net Hamiltonian can be any local operator which acts on quantum string states. Typically, the Hamiltonian can be divided into potential and kinetic energy pieces:

$$H = UH_U + hH_h + tH_t. \tag{11.2.1}$$

The constraint term UH_U with large U makes the ends of string to cost a lot of energy. So only closed strings exist at low energies. The kinetic energy H_t term describes the hopping (or the shape changing) of the closed string-nets, while the potential energy H_h is typically some kind of string tension. The string-net Hamiltonian (11.2.1), like the Ising model (11.1.1), may contain two phases. When $h \gg t$, the string tension dominates and we expect the ground state to be a state with a few small strings. On the other hand, when $t \gg h$, the kinetic energy dominates, and we expect the ground state is filled by many large fluctuating strings (see Fig. 11.1). This state is likely to be a string-net condensed state. This is why we expect a phase transition between the string-net condensed state and normal state at some t/h on the order of unity.

11.3 Topologically invariant string-net wave functions

Before constructing a Hamiltonian that has a string-net condensed ground state, we will first looking for a concrete description of the sting-net condensed state.

Let us try to visualize the wave function of a string-net condensed state. Recall that, according to our definition, the typical size of the string-nets in a string-net condensed state is on the order of the system size. (The motivation for this definition is that we want to distinguish string condensation from particle condensation. Indeed, if the string-nets were small compared with the system size, then in the long distance limit, we could effectively treat the strings as particles). The universal features of a string-net condensed phase are contained in the long distance character of the wave functions. Typically, two different string-net condensed states that belong to the same quantum phase will have different wave functions. However, by the standard RG reasoning, we expect that the two wave functions will look the same at long distances. That is, the string-net wave function for stings that only differ in short distance details, like those shown in Fig. 11.2, should be the same (or related). If we ignore those short-distance details, some long distance features of the wave functions are universal. It is these long distance features that describe different string-net condensed phases. Thus the key to understanding string-net condensed phases is to capture these universal long distance features.

Our approach for capturing the long distance universal properties of string-net states is to construct "fixed-point" wave functions that look the same at all length scales. If we could do an RG analysis on ground state wave functions, we would expect that all the states in a string condensed phase would flow

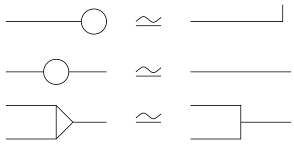


Figure 11.2: At long distances, a loop at the end, a bubble in a string, and a complicated branching are unobservable. So the amplitude for the corresponding string-net configurations are related.

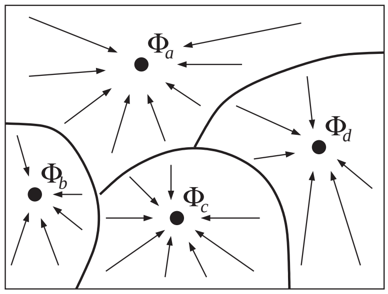


Figure 11.3: A schematic RG flow diagram for a string-net model with a few string-net condensed phases a, b, c, and d. All the states in each phase flow to fixed-points in the long distance limit. The corresponding fixed-point wave functions Φ_a, Φ_b, Φ_c, and Φ_d capture the universal long distance features of the associated quantum phases.

to the fixed point state (see Fig. 11.3). Thus, these wave functions are in one-to-one correspondence with string-net condensed phases.

Here we will restrict our attention to "fixed-point" wave functions that are topologically invariant. Those wave functions have the property that string-net configurations have the same amplitude if they can be continuously deformed into one another. Clearly the topologically invariant wave functions only depend on how the strings are connected and how they wrap around each other. The wave functions are invariant under the scaling transformation. These topologically invariant wave functions correspond to string-net condensed phases.

11.4 Describing 2D string-net condensation through local rules

One way to describe a string-net condensed wave function is to specify an amplitude for every string-net configuration. But a generic string-net condensed wave function is too complicated to be described this way. So instead, we will describe these topologically invariant wave functions indirectly – through local rules. The local rules are linear equations that relate the amplitudes of few string-net configurations which only differ locally from each other. We can then construct the string-net wave functions that satisfy these relations. If the local rules are complete enough, their uniquely determine a topological string-net wave function. So a topologically invariant string-net wave function can be specified by a proper set of local rules.

In the following, we will restrict ourselves to string-net condensations in two dimensions. To describe a set of local rules for 2D topological string-net wave functions, we need a set of data. Let us first describe

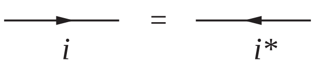


Figure 11.4: i and i^* label strings with opposite orientations.

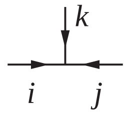


Figure 11.5: The orientation convention for the branching rules.

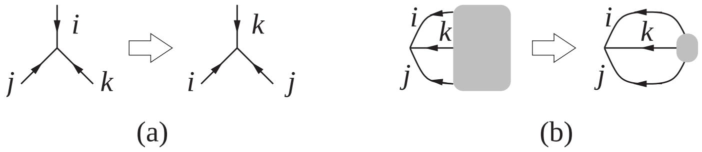


Figure 11.6: (a) If $\{i,j,k\}$ is an allowed branching, after rotating by $120°$, we see that $\{k,i,j\}$ is also an allowed branching. (b) The shaded area represents some arbitrary string-net configuration. If $\{i,j,k\}$ is an allowed branching, then after squeezing the rest of string-net into a small area and looking from far away, we see that $\{i^*,k^*,j^*\}$ is also an allowed branching.

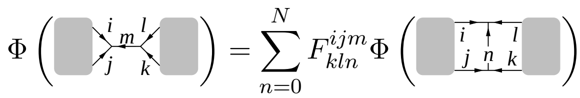
such set of data. Motivated by the fusion algebra in the conformal field theory[Moore and Seiberg (1989)], we may choose the following set of data to describe a set of local rules:

1. **Types of strings**: An integer N describing the number of different types of strings. The different types of strings will be labeled by $i = 1, ..., N$. In later discussions, we find that it is convenient to include $i = 0$ which corresponds to no-string (or a null string). We will call string labeled by i type-i string. Every label i has its dual label i^*, which satisfies $(i^*)^* = i$ and $0^* = 0$. i and i^* label strings with opposite orientations (see Fig. 11.4). If $i^* = i$, then we say the string is non-oriented.

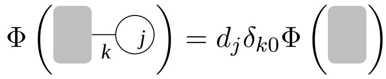
2. **Branching rules**: A collection of triplets $\{\{i,j,k\},\{l,m,n\}...\}$. These triplets correspond to the allowed branchings in the string-net theory. That is the amplitude of a string-net is non-zero in the string-net condensed state, if the string-net satisfies the branching rule. On the other hand, the amplitude of a string-net is zero in the string-net condensed state, if one of the branching points in the string-net does not satisfy the branching rule. The orientation convention for the branching rules is shown in Fig. 11.5. The set of the triplets has the property that if $\{i,j,k\}$ is in the set of allowed branching, then $\{k,i,j\}$ and $\{i^*,k^*,j^*\}$ are also in the set (see Fig. 11.6). Also $\{0,i,j\}$ is in the set if and only if $i = j^*$. This insures that a type-i string with no branching is an allowed string-net configuration. Since $0^* = 0$, so the $\{0,0,0\}$ is always an allowed branching. Using the branching rules, we can define a function δ_{ijk}: $\delta_{ijk} = 1$ if $\{i,j,k\}$ satisfies the branching rule (i.e. is in the set) and $\delta_{ijk} = 0$ otherwise.

3. $6j$ **symbols**: A rank-6 complex tensor F^{ijm}_{kln}.

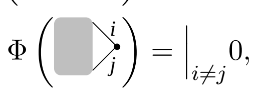
The data specify the following set of local rules:



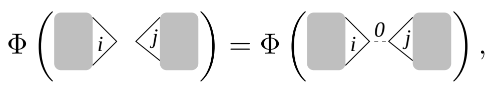
(11.4.1)



(11.4.2)



(11.4.3)



(11.4.4)

where $d_j \equiv 1/F^{jj^*0}_{jj^*0}$, i, j, k etc label the different strings (including the null-string) and the shaded areas represent some arbitrary string-net configurations. Note that we have already assumed that the string-net wave function is topologically invariant. So we do not care about the sizes and the shapes of the string-net. We only care how strings are connected and how they wind around each other. Since there is no scale dependence, the local rules describe a scale invariant string-net wave function. These scale invariant string-net wave functions are ideal representatives of different quantum phases of string-net condensed states. The universal features of the string-net condensed states are embedded in the local rules. We also like to remark that the branches $\{i, j, m\}$, $\{m^*, l, k\}$, ... in eqn (11.4.1) are arbitrary and do not have to satisfy the branching rule.

The local moves in eqns eqn (11.4.1) – eqn (11.4.4) can connect any two string-nets, which in turn relate the amplitudes of the two string-nets. Thus the local rules eqns (11.4.1) – (11.4.4) are complete which allows us to determine the whole string-net wave function.[2]

The rule (11.4.2) with $j = 0$ tells us that open strings are not allowed (i.e. a string-net configuration has a vanishing amplitude if the strings have open ends). When $j \neq 0$, the string-net in eqn (11.4.2) can still be regarded as containing an open end from long distance point of view. Thus the string-net is not allowed even when $j \neq 0$. The rule (11.4.3) tells us that the switching between different types of strings are not allowed. If such switching is allowed, then the strings in the wave function only appear in certain mixed form. In this case, we should relabel such mixed string as our basic string type. The rule (11.4.4) indicates that we can freely add null strings to a string-net without changing its amplitude.

The possibilities for F^{ijm}_{kln} are highly restricted. An arbitrary choice of F^{ijk}_{lmn} does not lead to a single valued string-net wave function. This is because two string-nets may be connected by two different sequences of local moves. We need to choose the tensors F^{ijk}_{lmn}, carefully so that different sequences of local moves produce the same results. Finding those tensors is the topic of tensor category theory [Turaev (1994)]. It was shown that only those F^{ijm}_{kln} that satisfy the pentagon identity

$$\sum_n F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr^*} F^{riq^*}_{mls^*} \tag{11.4.5}$$

describe single valued string-net wave function. Thus finding different solutions to the pentagon identity is equivalent to finding different "fixed point" string-net condensed states or different phases of string-net condensed states. This leads to a classification of the phases of string-net condensed states. Just like symmetry groups classify different particle condensed phases (i.e. different symmetry breaking phases), the solutions of eqn (11.4.5) classify different string-net condensed states.

It is a highly non-trivial exercise to find a set of self consistent local rules (i.e. solutions to the pentagon identity). It turns out that if we regard the index i that labels different types of the string as the index that labels different types of representations of a group G, then the $6j$ symbol of the group G provides a solution of the pentagon identity (after a proper rescaling). The low energy effective theory of the corresponding string-net condensed state turns out to be a gauge theory with G as the gauge group. Thus, gauge bosons and gauge group emerge from string-net condensation in a very natural way.

The string-net picture of the gauge theory allows us to understand why the low energy effective theories of certain systems are gauge theories, and why the system chooses a particular group as its gauge group. By changing the coupling constants of the system, the system may choose a different string-net condensed state described by a different $6j$ symbol as its ground state. The different $6j$ symbol will result in a different gauge group. This results in a phase transition that changes the gauge group of the low energy effective theory.

In 2+1 dimensions, some solutions of the pentagon identity do not correspond to the $6j$ symbols of groups, but rather the $6j$ symbols of quantum groups. In these cases, the low energy effective theories of the corresponding string-net condensed states are Chern-Simons gauge theories.

[2]This result is highly non-trivial. It is discussed in tensor category theory [Turaev (1994)].

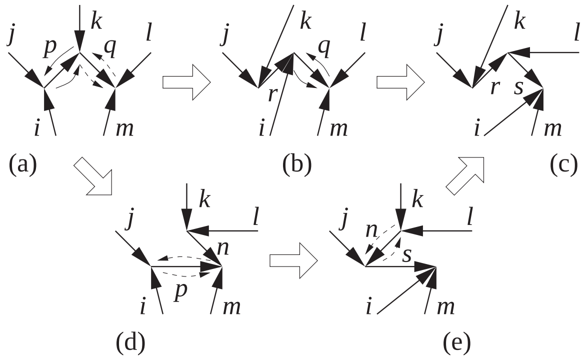


Figure 11.7: The local rules (11.4.1) are implemented by switching the legs according to the arrows. The operations (a) → (b) → (c) are done according to the solid arrows. The operations (a) → (d) → (e) → (c) are done according to the dashed arrows. The two sequences of the operation should lead to the same linear relations between the string-net configurations (a) and (c).

In general, different string-net condensed states correspond to states with different topological order. The corresponding low energy effective theories are different topological field theories. The classification of string-net condensation leads to a classification of topological field theories and classification of topological orders.

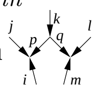
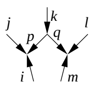
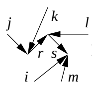
11.5 The pentagon identity

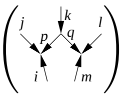
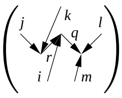
In the following, we will derive the the pentagon identity using a graphic method. We start with the string-net wave function described by a set of local rules (11.4.1) – (11.4.4). We assume that a string-net X will have a non-zero amplitude, $\Phi(X) \neq 0$, if all the branchings in X are allowed branchings. Rotating the string-net in eqn (11.4.1) by $180°$, we see that F^{ijm}_{kln} must satisfies

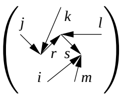
$$F^{ijm}_{kln} = F^{klm^*}_{ijn^*}.$$

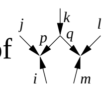
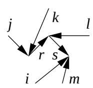
Also setting $i = l = n = 0$, $j = k^*$, and $m = k$ in eqn (11.4.1), we find

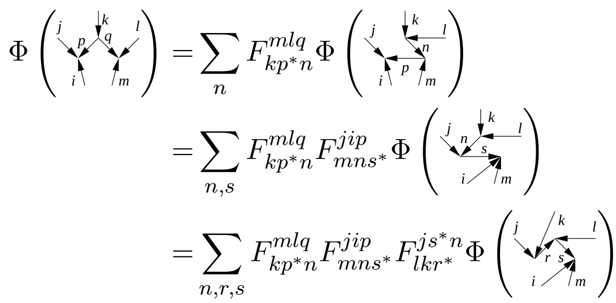
$$F^{0k^*k}_{k00} = 1 \tag{11.5.1}$$

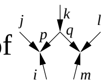
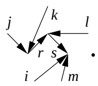
The $6j$ symbol F^{ijk}_{lmn} also satisfies other conditions. If we apply the local rules (11.4.1) twice on the string-net configuration  as shown in Fig. 11.7a, Fig. 11.7b, and Fig. 11.7c, we find that the amplitude of  is related to the amplitude of :

$\Phi\Big($  $\Big) = \sum_r F^{jip}_{q^*kr^*}\,\Phi\Big($  $\Big)$

$= \sum_{r,s} F^{jip}_{q^*kr^*} F^{riq^*}_{mls^*}\,\Phi\Big($  $\Big)$

We can find another relation between the amplitudes of  and  by applying the local rules three times as shown in Fig. 11.7a, Fig. 11.7d, Fig. 11.7e, and Fig. 11.7c:

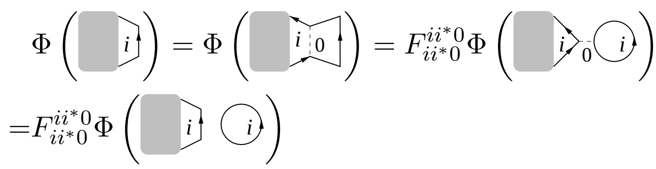


The two sequences of the local moves must result in the same relation between the amplitude of  and the amplitude of . Thus in order for the local rules to be self consistent, F^{ijk}_{lmn} must satisfy the pentagon identity (11.4.5). It turns out that eqn (11.4.5) is not only the necessary condition for the self consistent local rules, after supplemented with with a few minor conditions (see eqns (11.6.11), (11.6.12), and (11.6.14)), the pentagon identity (or its variant form (11.6.14)) is also the sufficient condition.

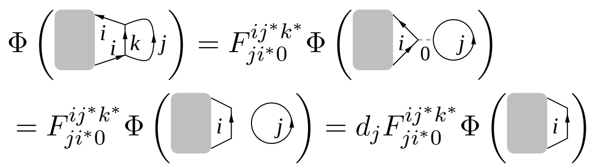
11.6 Properties of self consistent local rules

11.6.1 Simple properties of F^{ijk}_{lmn}

The local rules (11.4.1) – (11.4.4) allow us to obtain the relation between the amplitudes of string-net configurations. For example



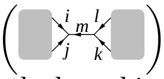
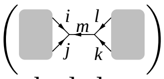
where we have used eqn (11.4.2). $F^{ii^*0}_{ii^*0}$ is an important quantity. If $F^{ii^*0}_{ii^*0} = 0$, from the above calculation, we see that the type-i string will not be allowed (i.e. any string-net containing the type-i string will have an vanishing amplitude). Thus we can assume $F^{ii^*0}_{ii^*0} \neq 0$ and $d_i = 1/F^{ii^*0}_{ii^*0}$ that we introduced in eqn (11.4.2) always exist. We note that $d_0 = 1$. Similarly, we can find



(11.6.1)

11.6.2 Rescaling of F^{ijk}_{lmn}

The solution of the pentagon identity is not unique. If F^{ijk}_{lmn} describes a self consistent local rule, we may obtain some other self consistent F^{ijk}_{lmn} by rescaling the wave function of the string-net. The rescaling is done by multiplying to the string-net wave function a factor $f(i,j,k)$ for each vertex $\{i,j,k\}$ in the string-net. Here $f(i,j,k)$ satisfies $f(i,j,k) = f(j,k,i)$ and $f(0,i,i^*) = 1$. For example, under the rescaling,

Φ  is changed to $f(i,j,m)f(k,l,m^*)(...)\Phi$ , where $(...)$ represents factors of f for the branching points in the string-net represented by the shaded area. The rescaling will cause a change in F:

$$F^{ijm}_{kln} \to \tilde{F}^{ijm}_{kln} = F^{ijm}_{kln} \frac{f(n,l,i)f(j,k,n^*)}{f(i,j,m)f(k,l,m^*)} \tag{11.6.2}$$

If F^{ijl}_{lmn} a solution of the pentigon identity (11.4.5), then the rescaled $\tilde{F}^{ijk}_{lmn}$ is also self consistent. Since the wave function and the rescaled wave function are related through a smooth deformation, the two wave functions describe quantum states in the same phase. For this reason, we regard F and $\tilde{F}$ to be equivalent.

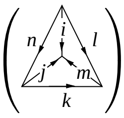
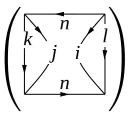
Some combinations of F^{ijk}_{lmn} are invariant under the rescaling transformation. Those invariant combinations will characterize the different equivalent classes of the self consistent local rules, or in another word, different phases of string-net condensed states. The simplest invariant combination is $F^{ss^*0}_{ss^*0}$. H^{hsg} defined below

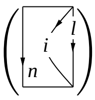
$$H^{hsg} \equiv \frac{F^{gg^*0}_{s^*sh} F^{hsg}_{s^*h^*0}}{F^{s^*s0}_{s^*s0}} = d_{s^*} F^{gg^*0}_{s^*sh} F^{hsg}_{s^*h^*0} \tag{11.6.3}$$

are also invariant combinations of F^{ijk}_{lmn}. In general, any relations between different string-net configurations that contain no branching points are invariant under the rescaling transformation. H^{hsg} is one such relation (see (11.9.7)).

11.6.3 Tetrahedral symmetry and symmetric $6j$ symbol G^{ijk}_{lmn}

F^{ijm}_{kln} has $(N+1)^6$ components, which is a large number. However, as a solution of the pentagon identity, many of those components are not independent. To see the relation between those components, let us consider

Φ  $= F^{ijm}_{kln}\Phi$ 

$= F^{ijm}_{kln} F^{nk^*j^*}_{kn^*0} d_k \Phi$ 

$$= F^{ijm}_{kln} F^{nk^*j^*}_{kn^*0} F^{n^*i^*l^*}_{in0} d_k d_i d_n \Phi(\emptyset) \tag{11.6.4}$$

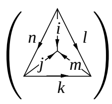
where we have used eqn (11.6.17) in the first line. We define the above combination in the front of $\Phi(\emptyset)$ as:

$$\tilde{G}^{ijm}_{kln} \equiv F^{ijm}_{kln} F^{nk^*j^*}_{kn^*0} F^{n^*i^*l^*}_{in0} d_k d_i d_n \tag{11.6.5}$$

Clearly $\tilde{G}^{ijm}_{kln}$ can be represented by a tetrahedron (see Fig. 11.8a).

$\tilde{G}^{ijm}_{kln}$ have the following properties:

1. $\tilde{G}^{ijm}_{kln} = 0$ unless all the branchings in the tetrahedron corresponding to F, ($\{i,j,m\}$, $\{i^*,l^*,n^*\}$, $\{j^*,n,k^*\}$, $\{k,l,m^*\}$), satisfy the branching rule. (See Fig. 11.8a).

2. Putting the tetrahedron shaped string-net on a sphere, we find that the amplitude Φ  has the tetrahedral symmetry. Therefore $\tilde{G}$ also has the tetrahedral symmetry. The tetrahedral symmetry is generated by two transformations (from Fig. 11.8a to Fig. 11.8b and Fig. 11.8a to Fig. 11.8c) and leads to:

$$\tilde{G}^{ijm}_{kln} = \tilde{G}^{klm^*}_{ijn^*} = \tilde{G}^{mij}_{nk^*l^*}$$

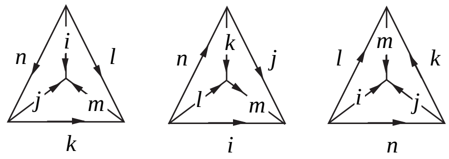


Figure 11.8: The three tetrahedrons are related by the tetrahedral symmetry. (a) The tetrahedron that represents $\tilde{G}^{ijm}_{kln}$ (or G^{ijm}_{kln}). (b) The tetrahedron obtained by rotating (a) around the axis connecting the centers of the link m and n by $180°$. Compare to (a) the orientation of the link m and n are reversed. Thus the tetrahedron (b) correspond to $\tilde{G}^{klm^*}_{ijn^*}$. (c) The tetrahedron obtained by rotating (a) around the center by $120°$. The orientations on the links are preserved. Thus the tetrahedron (c) corresponds to $\tilde{G}^{mij}_{nk^*l^*}$.

3. From the graphic representation of $\tilde{G}^{ijm}_{kln}$, we also see that

$$\tilde{G}^{ii^*0}_{ii^*0} = \tilde{G}^{000}_{ii^*i} = d_i$$

4. F^{ijm}_{kln} can be expressed in terms of $\tilde{G}^{ijm}_{kln}$:

$$F^{ijm}_{kln} \equiv \frac{d_n \tilde{G}^{ijm}_{kln}}{\tilde{G}^{nk^*j^*}_{kn^*0}\tilde{G}^{n^*i^*l^*}_{in0}} \tag{11.6.6}$$

To show (11.6.6), we first set $n = 0$, $l = i^*$, and $j = k^*$ in (11.6.5):

$$\tilde{G}^{ik^*m}_{ki^*0} = F^{ik^*m}_{ki^*0}F^{0k^*k}_{k00}F^{0i^*i}_{i00}d_k d_i d_0 = F^{ik^*m}_{ki^*0}d_k d_i$$

where we have used eqn (11.5.1). Expressing $F^{ik^*m}_{ki^*0}$ in terms of $\tilde{G}$, we can obtain (11.6.6) from (11.6.5). Eqn (11.6.6) allows us to see the relation between many components of F through the tetrahedral symmetry of $\tilde{G}$.

11.6.4 The complete self consistent equations

We have mentioned that in order for F^{ijk}_{lmn} to describes a string-net condensed state, F^{ijk}_{lmn} must satisfy the pentagon identity (11.4.5) and be consistent with the local rules (11.4.1) – (11.4.4). In the following, we would like to show that such F^{ijk}_{lmn} can be determined by a set of pure algebraic equations

$$\begin{aligned} F^{ijk}_{j^*i^*0} &= \frac{v_k}{v_i v_j}\delta_{ijk}, \\ F^{ijm}_{kln} &= F^{klm^*}_{ijn^*} = F^{mij}_{nk^*l^*}\frac{v_m v_n}{v_j v_l}, \\ \sum_n F^{mlq}_{kp^*n}F^{jip}_{mns^*}F^{js^*n}_{lkr^*} &= F^{jip}_{q^*kr^*}F^{riq^*}_{mls^*} \end{aligned} \tag{11.6.7}$$

By putting a single loop of type-i string on a sphere, we can continuously change it into a loop of the type-i^* string. This allows us to show that $d_i = d_{i^*}$. Thus we can introduce the weights of strings, v_i, that satisfy

$$v_i^2 = d_i, \quad v_i = v_{i^*}$$

This defines the v_i's in eqn (11.6.7).

We note that eqn (11.6.7) contains $(N+1)^3 + 2(N+1)^6 + (N+1)^8$ equations. Many of those equations are not independent. But there are enough independent equations to determine the $(N+1)^6$ components in F^{ijk}_{lmn}.

Under the rescaling transformation (11.6.2), $\Phi\left(\begin{smallmatrix} i\\ j\\ k\end{smallmatrix}\right)$ is changed to $f(i,j,k)f(i^*,k^*,j^*)\Phi\left(\begin{smallmatrix} i\\ j\\ k\end{smallmatrix}\right)$. Let us choose

$$f(i,j,k) = f(i^*,k^*,j^*) = \sqrt{\frac{v_i v_j v_k \Phi(\emptyset)}{\Phi\left(\begin{smallmatrix} i\\ j\\ k\end{smallmatrix}\right)}}, \tag{11.6.8}$$

for allowed branchings $\{i,j,k\}$. We note that $f(i,j,k)$ defined in (11.6.8) satisfies $f(i,i^*,0) = 1$ and $f(i,j,k) = f(j,k,i)$. Such a rescaling transformation will make

$$\Phi\left(\begin{smallmatrix} i\\ j\\ k\end{smallmatrix}\right) = v_i v_j v_k \delta_{ijk} \Phi(\emptyset)$$

This implies that

$$\tilde{G}^{ii^*0}_{k^*kj} = \tilde{G}^{ijk}_{j^*i^*0} = \frac{\Phi\left(\begin{smallmatrix} i\\ j\\ k\end{smallmatrix}\right)}{\Phi(\emptyset)} = v_i v_j v_k \delta_{ijk} \tag{11.6.9}$$

and

$$F^{ijm}_{kln} = \frac{\tilde{G}^{ijm}_{kln}}{v_i v_j v_k v_l} \tag{11.6.10}$$

It is more convenient to introduce symmetric $6j$ symbol

$$G^{ijm}_{kln} \equiv \frac{\tilde{G}^{ijm}_{kln}}{v_i v_j v_m v_k v_l v_n}$$

which have the same tetrahedral symmetry as $\tilde{G}^{ijm}_{kln}$:

$$G^{ijm}_{kln} = G^{klm^*}_{ijn^*} = G^{mij}_{nk^*l^*}. \tag{11.6.11}$$

In terms of G, (11.6.9) becomes the 1G relation

$$G^{ii^*0}_{k^*kj} = \frac{\delta_{ijk}}{v_i v_k}, \qquad G^{ijk}_{j^*i^*0} = \frac{\delta_{ijk}}{v_i v_j} \tag{11.6.12}$$

and (11.6.10) becomes

$$F^{ijm}_{kln} = G^{ijm}_{kln} v_m v_n \tag{11.6.13}$$

We see that the $6j$ symbol F can be expressed in terms of the the symmetric $6j$ symbol G and the weights v_i. The above relation and the graphic representation of $\tilde{G}^{ijk}_{lmn}$ in eqn (11.6.4) allow us to show that F^{ijk}_{lmn} is non-zero only if $\{i,j,m\}$, $\{m^*,k,l\}$, $\{i,l^*,n^*\}$, and $\{j,n,k^*\}$ satisfy the branching rule. Eqns (11.6.12) and (11.6.6) allow us to obtain the first line of eqn (11.6.7) and eqns (11.6.11) and (11.6.6) allow us to obtain the second line of eqn (11.6.7).

The pentagon identity (11.4.5) becomes

$$\sum_n d_n G^{mlq}_{kp^*n} G^{jip}_{mns^*} G^{js^*n}_{lkr^*} = G^{jip}_{q^*kr^*} G^{riq^*}_{mls^*} \tag{11.6.14}$$

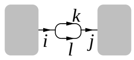
when written in terms of G. Setting $r = 0$, $s = l$, and $j = k^*$, we obtain a simpler 2G relation

$$\sum_n G^{mlq}_{kp^*n} G^{l^*m^*i^*}_{pk^*n} d_n = \frac{\delta_{iq}}{d_i} \delta_{mlq} \delta_{k^*ip} \tag{11.6.15}$$

Using eqn (11.6.12) and eqn (11.6.13), we can also show that

$$F^{ii^*0}_{j^*jk} F^{kji}_{j^*k^*0} d_{j^*} = \delta_{kji} \tag{11.6.16}$$

Problem 11.6.1 :
Show that

$$\Phi$$  $$= 0, \quad \text{if } i \neq j. \tag{11.6.17}$$

Problem 11.6.2 :
Show eqn (11.6.16).

11.7 Some simple examples of string-net condensed states

In this section, we will construct a few simple solutions of eqns. (11.6.11), (11.6.12) and (11.6.14) (which are equivalent to eqn (11.6.7)). As discuss above, each solution gives us a string-net condensed state.

11.7.1 The $N = 1$ closed-string condensed states

The simplest string-net condensed states are described by the following set of data:

1. **Types of strings:** There is only one type of string, $N = 1$. The string is not oriented $1^* = 1$.
2. **Branching rules:** The allowed branches are $\{0, 0, 0\}$, $\{1, 1, 0\}$, $\{1, 0, 1\}$, and $\{0, 1, 1\}$. The branching rules determines the function δ_{ijk}.

Since no branching is allowed, the above data describe $N = 1$ closed-string condensed state. The $N = 1$ string is alway non-oriented.

The solution of of eqns. (11.6.11), (11.6.12) and (11.6.14), G^{ijk}_{lmn}, can be found through the following steps:

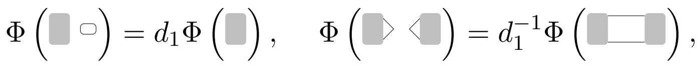
1. From the branching rules, we find that only the following G^{ijk}_{lmn}'s are non-zero. Those non-zero G^{ijk}_{lmn}'s are related by the tetrahedral symmetry:
G^{000}_{000},
$G^{000}_{111} = G^{110}_{001} = G^{011}_{100} = G^{101}_{010}$,
$G^{110}_{110} = G^{101}_{101} = G^{011}_{011}$.
2. From (11.6.12), we find
$G^{000}_{000} = 1$,
$G^{000}_{111} = G^{110}_{001} = G^{011}_{100} = G^{101}_{010} = \frac{1}{v_1}$,
$G^{110}_{110} = G^{101}_{101} = G^{011}_{011} = \frac{1}{v_1^2}$.
3. Setting $m = l = k = p = 1$ and $i = q = 0$ in (11.6.15), we find $\sum_n G^{110}_{11n} G^{110}_{11n} d_n d_0 = \delta_{00}\delta_{110}\delta_{101}$ or $G^{110}_{110} G^{110}_{110} = 1$. Since $G^{110}_{110} = \frac{1}{v_1^2}$, we find v_1 can take one of the four values 1, -1, i, and $-i$.

Now the values of all the components of G^{ijm}_{kln} are determined. We can check that the above symmetric $6j$ symbol does satisfy the full pentagon identity (11.6.14).

From eqn (11.6.13), we find that the above four solutions of G^{ijk}_{lmn} only give us two distinct F^{ijk}_{lmn}:

$$\begin{aligned} F^{000}_{000} &= F^{011}_{011} = F^{110}_{001} = F^{101}_{010} = 1, \\ F^{101}_{101} &= F^{000}_{111} = F^{011}_{100} = 1, \\ F^{110}_{110} &= d_1^{-1} \end{aligned} \tag{11.7.1}$$

with $d_1 = v_1^2 = \pm 1$, which describe two closed-string condensed states. The local rules (11.4.1) and (11.4.2) become



(11.7.2)

The local rules are so simple that we can calculate the corresponding closed-string wave functions explicitly. We find

$$\Phi(X) = d_1^{N_l},$$

where N_l is the number of the closed loops in the closed-string configuration X. As we will see later that the above two closed-string condensed states correspond to a Z_2 gauge theory and a $U(1) \times U(1)$ Chern-Simons theory with a K-matrix $K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. (The Z_2 gauge theory can also be viewed as a $U(1) \times U(1)$ Chern-Simons theory with a K-matrix $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$).

11.7.2 The $N = 1$ string-net condensed states

The next simplest string-net condensed states are described by

1. **Types of strings:** There is only one type of string, $N = 1$. The string is not oriented $1^* = 1$.
2. **Branching rules:** The allowed branches are $\{0,0,0\}$, $\{1,1,0\}$, $\{1,0,1\}$, $\{0,1,1\}$, and $\{1,1,1\}$.

Since branching is allowed, the above data describe $N = 1$ string-net condensed state. Again, the $N = 1$ string is alway non-oriented.

The solution of of eqns. (11.6.11), (11.6.12) and (11.6.14), G^{ijk}_{lmn}, can be found through the following steps:

1. From the branching rules, we find that only the following G^{ijk}_{lmn}'s are non-zero. Those non-zero G^{ijk}_{lmn}'s are related by the tetrahedral symmetry:
G^{000}_{000},
G^{111}_{111},
$G^{000}_{111} = G^{110}_{001} = G^{011}_{100} = G^{101}_{010}$,
$G^{110}_{110} = G^{101}_{101} = G^{011}_{011}$,
$G^{110}_{111} = G^{101}_{111} = G^{011}_{111} = G^{111}_{110} = G^{111}_{101} = G^{111}_{011}$.
2. From the 1G relations (11.6.12), we find
$G^{000}_{000} = 1$,
$G^{000}_{111} = G^{110}_{001} = G^{011}_{100} = G^{101}_{010} = \frac{1}{v_1}$,
$G^{110}_{110} = G^{101}_{101} = G^{011}_{011} = G^{110}_{111} = G^{101}_{111} = G^{011}_{111} = G^{111}_{110} = G^{111}_{101} = G^{111}_{011} = \frac{1}{v_1^2}$.
3. One of the 2G relations (11.6.15), $\sum_k G^{11k}_{110} G^{11k}_{110} d_k d_0 = \delta_{00}$, requires that $G^{110}_{110} G^{110}_{110} + G^{111}_{110} G^{111}_{110} d_1 = 1$. Since $G^{110}_{110} = G^{111}_{110} = \frac{1}{v_1^2}$, we find v_1 satisfies
$$v_1^4 - v_1^2 - 1 = 0. \tag{11.7.3}$$
Thus v_1 can only take one of the following four values
$$v_1 = \pm\sqrt{\frac{\sqrt{5}+1}{2}}, \qquad v_1 = \pm i\sqrt{\frac{\sqrt{5}-1}{2}}.$$
Another 2G relation, $\sum_k G^{11k}_{111} G^{11k}_{110} d_k d_0 = \delta_{10}$, requires that $G^{110}_{111} G^{110}_{110} + G^{111}_{111} G^{111}_{110} d_1 = 0$. We find $G^{111}_{111} = -\frac{1}{v_1^4}$.

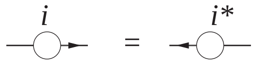


Figure 11.9: i on a link and i^* on the reverse link label the same spin state in the spin model.

Just like the $N = 1$ closed-string model, the 1G relations and the 2G relations again completely determine the values of the weights v_i and the $6j$ symbol G^{ijk}_{lmn}. We can check that the four $6j$ symbol G^{ijk}_{lmn} obtained this way are four solutions of the pentagon identity (11.6.14).

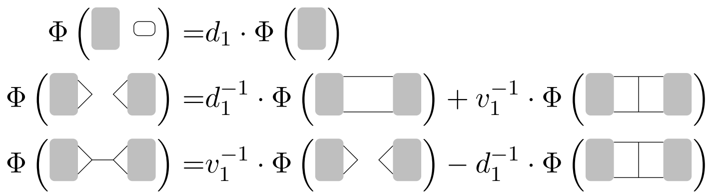
From eqn (11.6.13), we find that the above four solutions of G^{ijk}_{lmn} give us four distinct F^{ijk}_{lmn}'s:

$$
\begin{aligned}
&F^{000}_{000} = F^{000}_{111} = F^{110}_{001} = F^{101}_{010} = F^{011}_{100} = 1\\
&F^{101}_{101} = F^{011}_{011} = F^{111}_{011} = F^{011}_{111} = F^{111}_{101} = F^{101}_{111} = 1\\
&F^{110}_{111} = F^{111}_{110} = \frac{1}{v_1}\\
&F^{110}_{110} = -F^{111}_{111} = \frac{1}{v_1^2}
\end{aligned}
\tag{11.7.4}
$$

We note that for the above F^{ijm}_{kln}, the rescaling transformation (11.6.2) can change the sign of v_1: $v_1 \to -v_1$, if we choose $f(1,1,1) = i$ (note $f(1,1,0) = f(0,0,0) = 1$). Thus the F^{ijm}_{kln}'s obtained from v_1 with different signs lead to equivalent string-net condensed states. Thus v_1 has only two inequivalent choices

$$
v_1 = \sqrt{\frac{\sqrt{5}+1}{2}}, \qquad v_1 = i\sqrt{\frac{\sqrt{5}-1}{2}}.
$$

The local rules (11.4.1) and (11.4.2) become



(11.7.5)

Unlike the previous case, there is no closed form expression for the wave function amplitude.

11.8 Lattice "spin" models with string-net condensation

After constructed various string-net condensed wave functions, we would like to construct exactly soluble Hamiltonians such that the constructed string-net condensed wave functions are the ground state wave functions. In this case, we may say that the the systems described by the constructed Hamiltonians have string-net condensation.

However, to obtain a well behaved Hamiltonian, we need put the string-nets on a lattice. So in this section we will define a 2D lattice model that contain string-nets.

Our model is just the usual "spin" models with one spin on each link of a honeycomb lattice. Each spin has $N+1$ states labeled by $i = 0, 1, ..., N$. We will use $\boldsymbol{i}$, $\boldsymbol{j}$, ... to label different links (or different spins) and $\boldsymbol{I}$, $\boldsymbol{J}$, ... to label different vertices of the honeycomb lattice.

We assign each link an arbitrary direction (see Fig. 11.10). If the spin on a link are in a state labeled by i, then we say there is a type-i string in the link. The orientation of the string is in the direction of the link. Clearly, if we reverse the direction of the link, the label i will be change to label i^* in the model (see

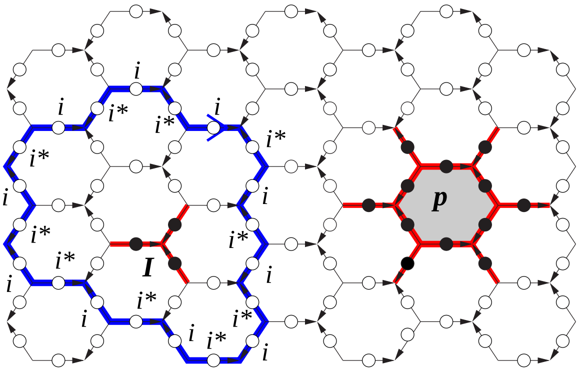


Figure 11.10: The honeycomb lattice with one site per link. A type-i closed string. In the single-spin model, $B^s_{\boldsymbol{p}}$ acts on the 12 sites represented by the solid circles around the hexagon $\boldsymbol{p}$. $E_{\boldsymbol{I}}$ acts on the three sites represented by the solid circles around the vertex $\boldsymbol{I}$.

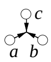
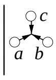
Fig. 11.9). The type-0 string is regarded as null string (i.e. no string on the link). A string-net configuration is formed by the non-trivial strings on the links. According to this definition, any spin configuration will correspond to a string-net configuration (see Fig. 11.10).

Now the question is that how to construct a Hamiltonian whose ground state is the string-net condensed state described in the last a few sections. We have seen that a string-net condensed state is described by a set of data: the branching rule δ_{ijk} and the $6j$ symbol F^{ijk}_{lmn}. So it is natural to expect that we can use the same set of data to construct a Hamiltonian whose ground state is the above string-net condensed state.

Although the string-net obtained from a generic spin configuration may not satisfies the branching rule, the string-nets in the string-net condensed state always satisfy the branching rule. So we want to construct a Hamiltonian whose ground state is formed by the string-nets that satisfy the branching rule. This can be achieved by including a term

$$U \sum_{\boldsymbol{I}} (1 - E_{\boldsymbol{I}})$$

in the spin Hamiltonian. The operator $E_{\boldsymbol{I}}$ is an operator that only act on the 3 sites that are next to the vertex $\boldsymbol{I}$. It is given by

$E_{\boldsymbol{I}} \Big|$  $\Big\rangle = \delta_{abc} \Big|$  $\Big\rangle$ (11.8.1)

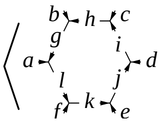
where δ_{abc} is the δ-symbol that describes the branching rule. Clearly, $E_{\boldsymbol{I}}$ and $1 - E_{\boldsymbol{I}}$ are projectors. When U is very large, the string-nets that do not satisfy the branching rule will have a energy of order at least U. In this case, the ground state (and other low energy excitations) with energies close to zero are only formed by string-nets that satisfy the branching rule. This way we implemented the branching rule in the ground state.

If the Hamiltonian only contain the $E_{\boldsymbol{I}}$ term, any string-net that satisfies the branching rule will be a ground state. In this case, a string-net cannot move and has no dynamics. Since there are many string-nets that satisfy the branching rule, the ground states are highly degenerate. We need to add additional terms to lift the degeneracy. The additional term will allow string-nets to fluctuate which will make the ground state to be a proper superposition of all string-nets that satisfy the branching rule. The desired Hamiltonian has a form

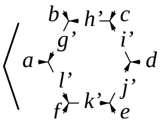
$$H_{\text{strnet}} = g \sum_{\boldsymbol{p}} (1 - B_{\boldsymbol{p}}) + U \sum_{\boldsymbol{I}} (1 - E_{\boldsymbol{I}}), \quad B_{\boldsymbol{p}} = \sum_{s=0}^{N} a_s B^s_{\boldsymbol{p}} \tag{11.8.2}$$

where $\sum_{\boldsymbol{p}}$ sum over all the hexagons of the honeycomb lattice and $\sum_{\boldsymbol{I}}$ sum over all the vertices of the honeycomb lattice.

Let us explain the terms in H_{strnet}. The hexagon operators $B^s_{\boldsymbol{p}}$ only act on the twelve spins on the six edges of the hexagon $\boldsymbol{p}$ and on the six legs of the hexagon $\boldsymbol{p}$ (see Fig. 11.10). Therefore, $B^s_{\boldsymbol{p}}$ correspond to a $(N+1)^{12} \times (N+1)^{12}$ matrix in the spin model. It turns out that the action of $B^s_{\boldsymbol{p}}$ does not change the spin states on the legs of the hexagon $\boldsymbol{p}$. Thus the above $(N+1)^{12} \times (N+1)^{12}$ matrix is block diagonalized. So at the end, $B^s_{\boldsymbol{p}}$ can be described by $(N+1)^6$ matrices. Each matrix is a $(N+1)^6 \times (N+1)^6$ matrix. Let us use $B^{s,ghijkl}_{\boldsymbol{p},mnopqr}(abcdef)$, with $a, ..., r = 0, 1, ..., N$, to denote the matrix elements of those $(N+1)^6$ matrices. We have



$$\Big\langle \cdots \Big| B^s_{\boldsymbol{p}} = \sum_{m,...,r} B^{s,ghijkl}_{\boldsymbol{p},g'h'i'j'k'l'}(abcdef) \Big\langle \cdots \Big| \tag{11.8.3}$$



Note the choice of the directions of the links which is different from that in Fig. 11.10. $B^{s,ghijkl}_{\boldsymbol{p},mnopqr}(abcdef)$ can be expressed in term of the $6j$ symbol F^{ijk}_{lmn}:

$$B^{s,ghijkl}_{\boldsymbol{p},g'h'i'j'k'l'}(abcdef) = F^{al^*g}_{s^*l'^*g'} F^{bg^*h}_{s^*g'^*h'} F^{ch^*i}_{s^*h'^*i'} F^{di^*j}_{s^*i'^*j'} F^{ej^*k}_{s^*j'^*k'} F^{fk^*l}_{s^*k'^*l'}. \tag{11.8.4}$$

We note that the resulting states after the action of $B^s_{\boldsymbol{p}}$ have branchings $\{b, g'^*, h'\}$ etc. Since $F^{bg^*h}_{s^*h'g'*} = 0$ if $\{b, g'^*, h'\}$ does not satisfies the branching rule, so the action of $B^s_{\boldsymbol{p}}$ always results in string-net states that satisfy the branching rule. Also if a state contain a branching, say $\{b, g^*, h\}$, that does not satisfy the branching rule, then the action of $B^s_{\boldsymbol{p}}$ will make such a state vanishes, since $F^{bg^*h}_{s^*h'g'*} = 0$ if $\{b, g^*, h\}$ does not satisfies the branching rule. Therefore,

$$B^s_{\boldsymbol{p}} = E_{\boldsymbol{I}} B^s_{\boldsymbol{p}} E_{\boldsymbol{I}}. \tag{11.8.5}$$

and $B^s_{\boldsymbol{p}}$ commute with $E_{\boldsymbol{I}}$ for any $\boldsymbol{p}$ and $\boldsymbol{I}$.

In the next section, we will show that the $B_{\boldsymbol{p}}$'s and $E_{\boldsymbol{I}}$'s all commute with each other. Thus our model H_{strnet} (11.8.2) is exactly soluble. If we also choose a_s to be

$$a_s = \frac{d_s}{\sum_{i=0}^{N} d_i^2}, \tag{11.8.6}$$

then

1. The $B_{\boldsymbol{p}}$'s in eqn (11.8.2) are projectors: $B^2_{\boldsymbol{p}} = B_{\boldsymbol{p}}$.
2. The ground state of H_{strnet} satisfies $B_{\boldsymbol{p}} = 1$ for all hexagons $\boldsymbol{p}$ and $E_{\boldsymbol{I}} = 1$ for all vertices $\boldsymbol{I}$.
3. The ground state is a string-net condensed state described by the set of local rules (11.4.1) – (11.4.4).

Since $(B_{\boldsymbol{p}}, E_{\boldsymbol{I}})$ is a set of commuting operators, we can choose the basis of the spin Hilbert space as the common eigenstates of $B_{\boldsymbol{p}}$ and $E_{\boldsymbol{I}}$:

$$B_{\boldsymbol{p}}|b_{\boldsymbol{p}}, e_{\boldsymbol{I}}, \alpha\rangle = b_{\boldsymbol{p}}|b_{\boldsymbol{p}}, e_{\boldsymbol{I}}, \alpha\rangle, \qquad E_{\boldsymbol{I}}|b_{\boldsymbol{p}}, e_{\boldsymbol{I}}, \alpha\rangle = e_{\boldsymbol{I}}|b_{\boldsymbol{p}}, e_{\boldsymbol{I}}, \alpha\rangle,$$

where $e_{\boldsymbol{I}} = 0, 1$ and $b_{\boldsymbol{p}} = b_0, b_1, ..., b_{M-1}$ are eigenvalues of $B_{\boldsymbol{p}}$. The index α labels the possible degenerate states: $\alpha = 1, 2, ..., \alpha_{\text{max}}$. In general α_{max} is a function of $\{b_{\boldsymbol{p}}, e_{\boldsymbol{I}}\}$. We can always choose b_0 to be the largest eigenvalue and rescale a_s to make $b_0 = 1$. Since the Hamiltonian H_{strnet} is a sum of $B_{\boldsymbol{p}}$'s and $E_{\boldsymbol{I}}$'s, the state $|b_{\boldsymbol{p}}, e_{\boldsymbol{I}}, \alpha\rangle$ is an energy eigenstate with energy $U \sum_{\boldsymbol{I}}(1 - e_{\boldsymbol{I}}) + g \sum_{\boldsymbol{p}}(1 - b_{\boldsymbol{p}})$. In particular, the ground state $|\Phi\rangle$ has zero energy and satisfies $E_{\boldsymbol{I}}|\Phi\rangle = B_{\boldsymbol{p}}|\Phi\rangle = |\Phi\rangle$.

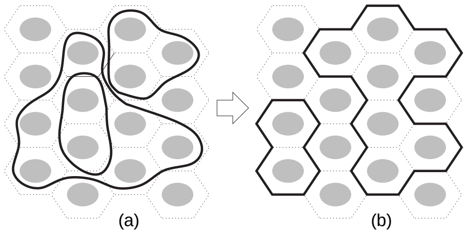


Figure 11.11: The fattened honeycomb lattice. The strings are forbidden in the shaded region. A string state in the fattened honeycomb lattice (a) can be viewed as a superposition of the string states on the links (b).

It is interesting to note that if $\alpha_{\text{max}} = 1$, The exact soluble model H_{strnet} can be mapped into a simple spin model with one spin-1/2 spin on each vertex and one spin-$(M-1)/2$ spin on each hexagon. The Hamiltonian of the dual spin model is

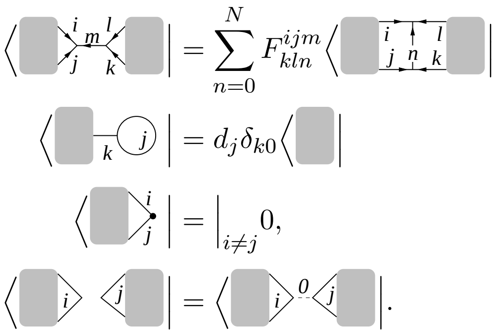
$$H_{\text{dual}} = U \sum_{\boldsymbol{I}} (1 + \sigma^z_{\boldsymbol{I}}) + g \sum_{\boldsymbol{p}} (1 + S_{\boldsymbol{p}})$$

where $S_{\boldsymbol{p}}$ is an $M \times M$ diagonal matrix with eigenvalues $b_0, ..., b_{M-1}$ which acts on the spin state on the hexagon $\boldsymbol{p}$.

11.9 Understanding lattice results using continuum string-nets

The above results look complicated and mysterious. One may wonder, how can one guess such complicated results. It turns out that there is a simple way to understand that above results. The string-net picture for two dimensional continuous space and the associated local rules (11.4.1) – (11.4.4) play a key role here.

We start with the 2D honeycomb lattice. We fatten the links into stripes of finite width (see Fig. 11.11). The key point here is that any continuum string-net state on the fattened honeycomb lattice (see Fig. 11.11a) can be viewed as a superposition of the string state with strings on the link (see Fig. 11.11b). This is because the string-net wave function $\Phi(X)$ for a string-net state is given by $\Phi(X) = \langle X|\Phi\rangle$, where $|X\rangle$ is the string-net state of a particular string-net configuration X. So the local rules (11.4.1) – (11.4.4) on the string-net wave function can be formally interpreted as a relation on the string-net states $\langle X|$:



(11.9.1)

(11.9.2)

(11.9.3)

(11.9.4)

Using the above local rules on the string-nets states, we can always write the state in Fig. 11.11a as a linear combination of the states in Fig. 11.11b. The physical states really correspond to string-nets on the links (such as the one in Fig. 11.11b). Drawing string-nets in the fattened honeycomb lattice is just a fancy way to represent physical string-net states. Every string-nets in the fattened honeycomb lattice (such as Fig. 11.11a) correspond to a superposition of the physical string-net states with strings on the links (such as Fig. 11.11b).

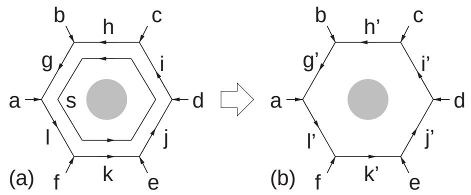
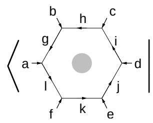
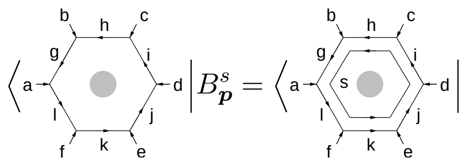
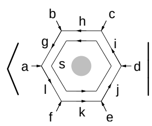


Figure 11.12: (a) The action of $B^s_{\boldsymbol{p}}$ can be represented by a loop of the type-s string. The string-net state (a) is actually a linear combination of the string-net states (b). The coefficient of the linear combination are obtained by the local rules (11.4.1) that change (a) to (b). Note that the center of the hexagon is the forbidden region. The local rules only apply to the ring-like region around the hexagon.

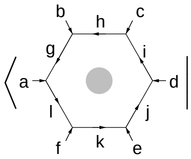
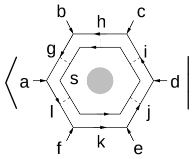
11.9.1 Graphic representation of the $B^s_{\boldsymbol{p}}$ operator

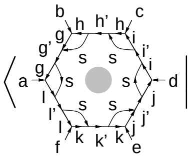
With this understanding, the $B^s_{\boldsymbol{p}}$ operator in (11.8.3) has a simple a graphic representation in the fattened honeycomb lattice. The operator $B^s_{\boldsymbol{p}}$ when acts on a string-net state  simply add a closed loop of type-s string:



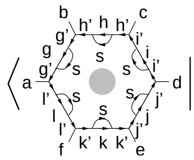
We can use the local rules (11.9.1) – (11.9.4) to write  as a linear combination of the physical string-net states with strings only on the links, i.e. to change Fig. 11.12a to Fig. 11.12b. This allows us to obtain the matrix elements of $B^s_{\boldsymbol{p}}$. The pentagon identity (11.4.5) ensures that different ways of change lead to identical result.

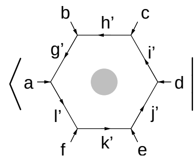
The following is a particular way to implement the above procedure.

 $B^s_{\boldsymbol{p}} =$  (11.9.5)

$$= \sum_{g'h'i'j'k'l'} F^{gg^*0}_{s^*sg'^*} F^{hh^*0}_{s^*sh'^*} F^{ii^*0}_{s^*si'^*} F^{jj^*0}_{s^*sj'^*} F^{kk^*0}_{s^*sk'^*} F^{ll^*0}_{s^*sl'^*}$$ 

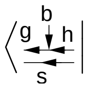
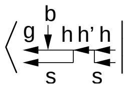
$$= \sum_{g'h'i'j'k'l'} F^{gg^*0}_{s^*sg'^*} F^{hh^*0}_{s^*sh'^*} F^{ii^*0}_{s^*si'^*} F^{jj^*0}_{s^*sj'^*} F^{kk^*0}_{s^*sk'^*} F^{ll^*0}_{s^*sl'^*} \times$$

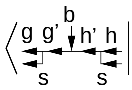
$$F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*}$$ 

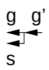
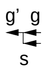
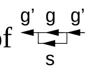
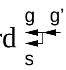
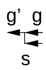
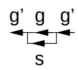
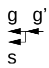
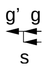
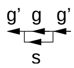
$$= \sum_{g'h'i'j'k'l'} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*}$$ 

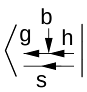
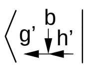
where we have used the local rules (11.9.1) and (11.6.1) in our calculations. We also used relation (11.6.16). (11.9.5) is exactly (11.8.4).

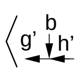
The above calculation can also be done locally near a vertex:

 $= \sum_{g'} F^{hh^*0}_{s^*sh'^*}$ 

$= \sum_{g'h'} F^{hh^*0}_{s^*sh'^*} F^{bg^*h}_{s^*h'g'^*}$ 

Since  and  always appears as part of  in the loop, we can regard  as 1 and  as , or $\frac{F^{g'^*sg}_{s^*g'0}}{F^{s^*s0}_{s^*s0}}$. This way a pair of  and  in a loop will reproduce . So we can rewrite the above as

 $= \sum_{g'h'} F^{hh^*0}_{s^*sh'^*} F^{bg^*h}_{s^*h'g'^*} \frac{F^{h'^*sh}_{s^*h'0}}{F^{s^*s0}_{s^*s0}}$ 

$= \sum_{g'h'} F^{bg^*h}_{s^*h'g'^*}$  (11.9.6)

which reproduces eqn (11.9.5).

11.9.2 Commuting properties of $B^{s_1}_{\boldsymbol{p}_1}$ and $B^{s_2}_{\boldsymbol{p}_2}$

Using the graphic representation of $B^s_{\boldsymbol{p}}$, we can easily show that $B^{s_1}_{\boldsymbol{p}_1}$ and $B^{s_2}_{\boldsymbol{p}_2}$ commute. Assume $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are next neighbor hexagons. The action of $B^{s_1}_{\boldsymbol{p}_1} B^{s_2}_{\boldsymbol{p}_2}$ on the string-net state Fig. 11.13a can be expressed as

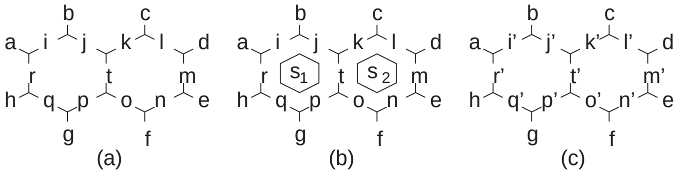


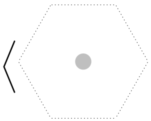
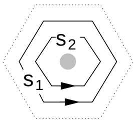
Figure 11.13: The action of $B^{s_1}_{\boldsymbol{p}_1}B^{s_2}_{\boldsymbol{p}_2}$ on the string-net state (a) can be represented by two loops of type-s_1 and type-s_2 strings which lead to (b). The string-net state (b) is a linear combination of the string-net state (c). The coefficients are obtained by the local rules that change (b) to (c).

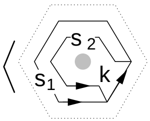
Fig. 11.13b. Fig. 11.13b is actually a linear combination of the string-net state Fig. 11.13c. The coefficients of the linear combination is determined by the local rules that changes Fig. 11.13b to Fig. 11.13c. Those coefficients are the matrix elements of $B^{s_1}_{\boldsymbol{p}_1}B^{s_2}_{\boldsymbol{p}_2}$. It is clear that, since the two loops of the type-s_1 and type-s_2 strings do not overlap, the action of $B^{s_2}_{\boldsymbol{p}_2}B^{s_1}_{\boldsymbol{p}_1}$ is represented by the same graph Fig. 11.13b. Thus $B^{s_2}_{\boldsymbol{p}_2}B^{s_1}_{\boldsymbol{p}_1}$ has the same matrix elements as $B^{s_1}_{\boldsymbol{p}_1}B^{s_2}_{\boldsymbol{p}_2}$.

We would like to remark that we did not assume the branching points to satisfy the branching rule in the above graphic calculation. Thus we have showed that $[B^{s_1}_{\boldsymbol{p}_1}, B^{s_2}_{\boldsymbol{p}_2}] = 0$ in the total Hilbert space of the spin model.

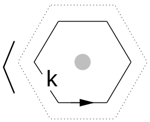
11.9.3 The condition for $B_{\boldsymbol{p}}$ to be a projector

To find the condition for $B_{\boldsymbol{p}}$ to be a projector, we consider

$\Big\langle$  $\Big| B^{s_1}_{\boldsymbol{p}} B^{s_2}_{\boldsymbol{p}} = \Big\langle$ 

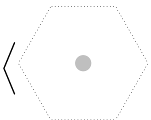
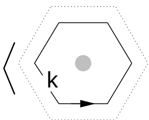
$\Big| = \sum_k F^{s_1 s_1^* 0}_{s_2^* s_2 k^*} \Big\langle$ 

$\Big|$

$= \sum_k F^{s_1 s_1^* 0}_{s_2^* s_2 k^*} F^{k^* s_2 s_1}_{s_2^* k 0} d_{s_2^*} \Big\langle$ 

$\Big|$ (11.9.7)

Using eqn (11.6.16), we find

$\Big\langle$  $\Big| B^{s_1}_{\boldsymbol{p}} B^{s_2}_{\boldsymbol{p}} = \sum_k \delta_{k^* s_2 s_1} \Big\langle$ 

$\Big|$

or

$$B^{s_1}_{\boldsymbol{p}} B^{s_2}_{\boldsymbol{p}} = \sum_k \delta_{k^* s_2 s_1} B^k_{\boldsymbol{p}}. \tag{11.9.8}$$

Let $B_{\boldsymbol{p}} = \sum_s a_s B^s_{\boldsymbol{p}}$. Using (11.9.8), we find that

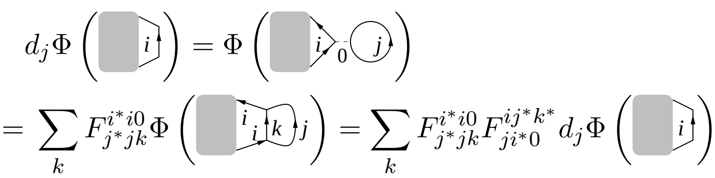
$$(B_{\boldsymbol{p}})^2 = \sum_{k,s_1,s_2} \delta_{k^* s_2 s_1} a_{s_1} a_{s_2} B^k_{\boldsymbol{p}}$$

So, if we choose a_s to satisfy

$$a_s = \sum_{s_1,s_2} \delta_{s^* s_2 s_1} a_{s_1} a_{s_2}, \tag{11.9.9}$$

then $B_{\boldsymbol{p}}$ will become a projection operator.

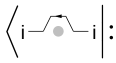
From

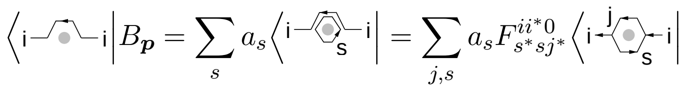


(11.9.10)

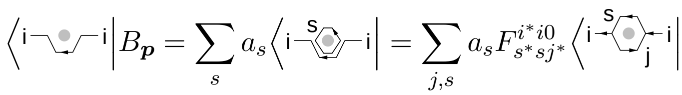
we can show that $\sum_k F^{i^*i0}_{j^*jk} F^{ij^*k^*}_{ji^*0} = 1$. Using eqn (11.6.7), we can reduce the above to $d_i = \sum_k \delta_{i^*kj} d_k/d_j$. This allows us to show $d_i \sum_j d_j^2 = \sum_k \delta_{i^*kj} d_k d_j$. We find that $a_s = d_s/\sum_j d_j^2$ is a solution of eqn (11.9.9).

11.9.4 The condition for the continuum string-net state

Now let us consider how $B_{\boldsymbol{p}}$ act on the state :



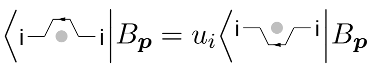
where the shaded area is the forbidden area in the center of the hexagon for applying the local rules. Similarly, for a state where the string going around the hexagon in the other way, we have



Thus if

$$a_s F^{ii^*0}_{s^*sj^*} = u_i a_j F^{i^*i0}_{j^*js^*} \tag{11.9.11}$$

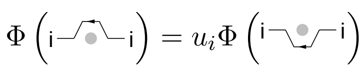
then



The condition (11.9.11) can be simplified to

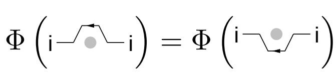
$$\frac{a_s}{a_j}\delta_{ij^*s} = u_i \frac{d_s}{d_j}\delta_{ij^*s} \tag{11.9.12}$$

Since the ground state of the string-net model H_{strnet} (11.8.2) is an eigenstate of $B_{\boldsymbol{p}}$, the ground state wave function Φ satisfies



We like to point out that the condition (11.9.11) or (11.9.12) is very important. Very often it is the condition for the Hamiltonian H_{strnet} to have a finite number of ground states.

A solution of eqn (11.9.9), $a_s = d_s/\sum_j d_j^2$, satisfies (11.9.11) with $u_i = 1$. In general if a_s satisfy eqn (11.9.12) with $u_i = 1$, the ground state of the string-net model H_{strnet} will satisfy



The forbidden region in the center of the hexagon will become unobservable and we can apply the local rules anywhere. The wave function satisfies eqn (11.4.1) – (11.4.4) on the hexagon lattice without any forbidden region. In this case, we say that the lattice model H_{strnet} has a canonical continuum limit and the lattice string-net wave function can be treated as a continuum string-net wave function.

A generic choice of a_s also give us exactly soluble models. But the lattice string-net wave function may not have a simple continuum limit due to the possible presence of fast oscillation in the string-net wave function at lattice scale.

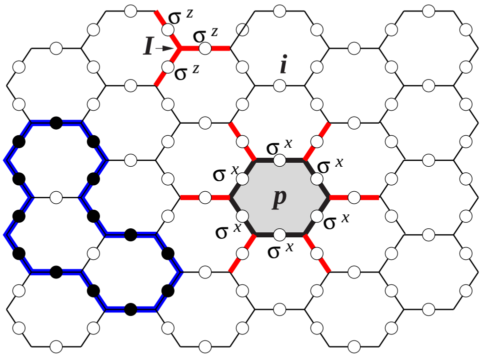


Figure 11.14: The Kagome lattice formed by spin-1/2 spins. The sites of the Kagome lattice are label by $\boldsymbol{i}$. The vertices, labeled by $\boldsymbol{I}$, form the honeycomb lattice. The open circles represent down-spins and the solid circles represent up-spins. The strings on the links are formed by up-spins. In this figure, the up-spins form a closed loop. The six think links that form the loop around the shaded hexagon is the edges of hexagon. The six thick links that are attached to the shaded hexagon are the legs of the hexagon. The three thick links connecting to the vertex $\boldsymbol{I}$ form the legs of $\boldsymbol{I}$. The U-term and the g-term in (11.10.4) are also presented in the figure.

11.10 Simple examples of exact soluble models

We have shown that each solution F^{ijk}_{lmn} of the self consistent equation (11.6.7) give rise to a string-net condensed state on the 2D honeycomb lattice and an exactly soluble Hamiltonian H_{strnet} (11.8.2) whose ground state is the string-net condensed state. We have given some simple examples of the solutions of eqn (11.6.7). In this section, we will use those simple solutions to construct exactly soluble string-net model on honeycomb lattice.

11.10.1 Exact soluble Hamiltonian for $N = 1$ closed-strings

The simplest solution (11.7.1) describes $N = 1$ closed strings. To use such a F^{ijk}_{lmn} to construct an explicit Hamiltonian, we need examins carefully the matrix elements of the $B^s_{\boldsymbol{p}}$ operator in eqn (11.9.5) or eqn (11.9.6). Since each spin has two states 0 and 1, the string-net model can be viewed as a spin-1/2 model on a Kagome lattice, if we identify the 0 state as a spin-down and the 1 state as a spin-up state (see Fig. 11.14). We find that

1. When it acts on a string-net state that does not satisfy the branching rule, $B^s_{\boldsymbol{p}}$ make the state vanishes.
2. $B^0_{\boldsymbol{p}}$ is determined by $F^{bg^*h}_{0h'g'^*}$ and $B^1_{\boldsymbol{p}}$ by $F^{bg^*h}_{1h'g'^*}$.
3. When it acts on a string-net state that satisfies the branching rule, $B^0_{\boldsymbol{p}}$ just do nothing.
4. When it acts on a string-net state that satisfies the branching rule, $B^1_{\boldsymbol{p}}$ flips the spins on the six edges of the hexagon $\boldsymbol{p}$ and multiply the resulting state with a phase.
5. The above phase is given by v_1^{-n} where n is the number of the up-spins on the six legs of the hexagon $\boldsymbol{p}$ (see Fig. 11.14).

In the spin-1/2 model, the operator $E_{\boldsymbol{I}}$ is given by

$$E_{\boldsymbol{I}} = \frac{1}{2} - \frac{1}{2} \prod_{\text{legs of } \boldsymbol{I}} \sigma^z_{\boldsymbol{k}} \tag{11.10.1}$$

The action of $B^0_{\boldsymbol{p}}$ can be produced by

$$B^0_{\boldsymbol{p}} = \prod_{\text{vertices of } \boldsymbol{p}} E_{\boldsymbol{I}} \tag{11.10.2}$$

and the action of $B^1_{\boldsymbol{p}}$ can be produced by

$$B^1_{\boldsymbol{p}} = \prod_{\text{legs of } \boldsymbol{p}} v_1^{-(1+\sigma^z_{\boldsymbol{i}})/2} \prod_{\text{edges of } \boldsymbol{p}} \sigma^x_{\boldsymbol{j}} \prod_{\text{vertices of } \boldsymbol{p}} E_{\boldsymbol{I}}, \tag{11.10.3}$$

where $\prod_{\text{legs}} v_1^{-(1+\sigma^z_{\boldsymbol{i}})/2}$ is the product of the spin operators on the six legs of the hexagon $\boldsymbol{p}$ and $\prod_{\text{edges}} \sigma^x_{\boldsymbol{j}}$ is the product of the spin operators on the six edges of the hexagon $\boldsymbol{p}$. So we can write the Hamiltonian H_{strnet} (11.8.2) explicitly in terms of spin-1/2 operators.

In the following we assume U is positive and very large. In this case the U-term in the Hamiltonian H_{strnet} enforces the branching rules When $g = 0$, the ground states of $H = U\sum_{\boldsymbol{I}} E_{\boldsymbol{I}}$ have an vanishing energy. The ground state is highly degenerate. Any string configurations that satisfy the branching rules correspond to the ground states. Those states are actually the close string states. All other string configurations that do not satisfy the branching rules (i.e. the states containing open strings) have energies of order U above the ground state. The g-term in H_{strnet} lifts the degeneracy and determines the dynamics of the strings.

Since $E_{\boldsymbol{I}}$ for the low energy excitations, we may set $E_{\boldsymbol{I}}$ in the g term to 1. After dropping the constant terms, we obtain the following $N = 1$ closed-string Hamiltonian

$$\begin{aligned} H =& U\sum_{\boldsymbol{I}} (\frac{1}{2} + \frac{1}{2} \prod_{\text{legs of } \boldsymbol{I}} \sigma^z_{\boldsymbol{k}}) - \frac{1}{2} g \sum_{\boldsymbol{p}} (\tilde{B}^1_{\boldsymbol{p}} + h.c.) \\ \tilde{B}^1_{\boldsymbol{p}} =& \prod_{\text{legs of } \boldsymbol{p}} v_1^{-(1+\sigma^z_{\boldsymbol{i}})/2} \prod_{\text{edges of } \boldsymbol{p}} \sigma^x_{\boldsymbol{j}} \end{aligned} \tag{11.10.4}$$

where $\sum_{\boldsymbol{p}}$ is the sum over all the hexagons of the honeycomb lattice and $\sum_{\boldsymbol{I}}$ is the sum over all the vertices of the honeycomb lattice. Eqn (11.8.2) and eqn (11.10.4) have the same low energy spectrum.

In the closed-string subspace

$$\tilde{B}^1_{\boldsymbol{p}} = (\tilde{B}^1_{\boldsymbol{p}})^\dagger, \qquad\qquad [\tilde{B}^1_{\boldsymbol{p}}, \tilde{B}^1_{\boldsymbol{p}'}] = 0. \tag{11.10.5}$$

All the low energy eigenstates for the model (11.10.4) are labeled by the common eigenvalues $\tilde{b}_{\boldsymbol{p}}$ of $\tilde{B}^1_{\boldsymbol{p}}$. The energy of the state is given by $-g\sum_{\boldsymbol{p}} \tilde{b}_{\boldsymbol{p}}$. $\tilde{b}_{\boldsymbol{p}} = \pm 1$ since $(\tilde{B}^1_{\boldsymbol{p}})^2 = 1$. Depending on the signs of g, and v_1^2, the model (11.10.4) can have four different ground states given by $\tilde{b}_{\boldsymbol{I}} = \text{sgn}(g)$. The ground states are superpositions of closed strings.

The excitations above the ground state are created by simply flipping the signs of a few $\tilde{b}_{\boldsymbol{p}}$'s and changing a few $E_{\boldsymbol{I}}$'s from 1 to 0. Those excitations are particles with short range interactions.

When $v_1 = 1$, eqn (11.10.4) becomes

$$H = U\sum_{\boldsymbol{I}} (\frac{1}{2} + \frac{1}{2} \prod_{\text{legs of } \boldsymbol{I}} \sigma^z_{\boldsymbol{k}}) - g\sum_{\boldsymbol{p}} \prod_{\text{edges of } \boldsymbol{p}} \sigma^x_{\boldsymbol{j}} \tag{11.10.6}$$

In the $U \to +\infty$ limit, the above Hamiltonian becomes the standard Hamiltonian for a Z_2 gauge theory on the honeycomb lattice. The excitations created by flipping the signs of $\tilde{b}_{\boldsymbol{p}}$'s correspond to the Z_2 vortex excitation in the Z_2 gauge theory. The excitations created by changing $E_{\boldsymbol{I}}$'s from 1 to 0 correspond to Z_2 charge excitations. Since $E_{\boldsymbol{I}} = 0$ only at the end of an open string, the Z_2 charge excitations correspond to ends of open strings.

The closed-string wave function for the ground state has a form

$$\Phi(X) = (\mathrm{sgn}(g))^{N_p}$$

where X represent closed-string configuration and N_p is the total number of the hexagons enclosed by the closed-strings in X. When $g > 0$, $\Phi(X)$ is simply the equal weight superposition of all closed strings. Such a closed-string condensed wave function have a nice continuum limit. When $g < 0$, the sign of $\Phi(X)$ changes at lattice scale and $\Phi(X)$ does not have a simple continuum limit. Such a state correspond to a state in Z_2 gauge theory with π-flux through each hexagon.

When $v_1 = i$, the ground state still correspond to closed-string condensed state. The wave function is given by

$$\Phi(X) = (\mathrm{sgn}(g))^{N_p}(-)^{N_l}$$

where N_l is the total number of the closed strings (the loops) in X.

11.10.2 $N = 1$ string-net models

The self consistent equation (11.6.7) also has the following two solution:

$$\begin{aligned}
&F^{000}_{000} = F^{110}_{001} = F^{101}_{010} = F^{011}_{011} = F^{111}_{011} = \\
&F^{000}_{111} = F^{011}_{100} = F^{101}_{101} = F^{011}_{111} = F^{111}_{101} = F^{101}_{111} = 1 \\
&F^{110}_{111} = F^{111}_{110} = \frac{1}{v_1} \\
&F^{110}_{110} = F^{111}_{111} = \frac{1}{v_1^2}
\end{aligned} \qquad (11.10.7)$$

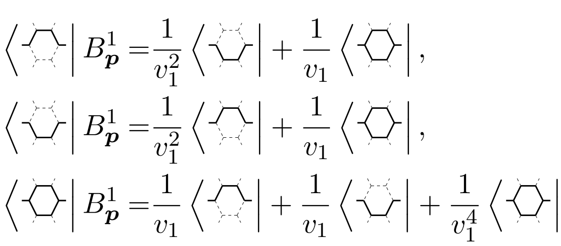
for the $N = 1$ non-oriented string-net. Here

$$v_1 = \sqrt{\frac{\sqrt{5}+1}{2}}, \ \ i\sqrt{\frac{\sqrt{5}-1}{2}}$$

describing the two solutions. The allowed branchings are $ijk = 000$, 110, 101, 011, and 111. So the string-net condensed state the above solution is a superposition of branched string-nets.

After obtaining F's and v's, we can construct the corresponding exact soluble Hamiltonian H_{strnet} (11.8.2). First we calculate the action of $B^1_{\boldsymbol{p}}$ on a few simple string-net states:

(11.10.8)



(11.10.9)

We find that B^1 and $B_{\boldsymbol{p}}$ is not hermitian when v_1 is not real. Thus only the solution with $v_1 = \sqrt{\frac{\sqrt{5}+1}{2}}$ leads to a physical string-net model.

To construct the exact soluble Hamiltonian H_{strnet} with $B_{\boldsymbol{p}}$ as a projector, we need solve (11.9.9) first. Eqn (11.9.9) has two solutions: $(a_0, a_1) = (\frac{5+\sqrt{5}}{10}, -\frac{1}{\sqrt{5}})$ and $(\frac{5-\sqrt{5}}{10}, \frac{1}{\sqrt{5}})$. The two sets of (a_0, a_1) allow us to construct two exactly soluble Hamiltonians H_{strnet}. The Hamiltonian from the first solution has extensive degenerate ground states. In fact the ground state degeneracy is given by F_{N_p} for an open lattice, where N_p is the number of the hexagons and F_n the Fubinachi number. The Hamiltonian has a finite number of degenerate ground states for the second solution. The second solution also satisfies (11.9.12) with $u_i = 1$. So the corresponding Hamiltonian has a canonical continuum limit and the string-net condensation on the lattice can be regarded as a string-net condensation in the continuum space. In the following we will concentrate on such a model. We call such a model $N = 1$ string-net model.

Again we can associate a down-spin with a null string and a up-spin with the type-1 string and write the Hamiltonian in terms of spin operators. So H_{strnet} is still a spin-1/2 model in Kagome lattice. However, its expression in terms of the spin-1/2 Pauli matrices is quit complicated. We will not write H_{strnet} in terms of the Pauli matrices.

All the eigenstates for the exactly soluble model are labeled by the common eigenvalues $(e_{\boldsymbol{I}}, b_{\boldsymbol{p}})$ of $E_{\boldsymbol{I}}$ and $B_{\boldsymbol{p}}$. The energy of the state is given by $\sum_{\boldsymbol{I}}(1 - e_{\boldsymbol{I}}) + \sum_{\boldsymbol{p}}(1 - b_{\boldsymbol{p}})$. $e_{\boldsymbol{I}} = 0, 1$ and $b_{\boldsymbol{p}} = 0, 1$ since both $E_{\boldsymbol{I}}$ and $B_{\boldsymbol{p}}$ are projection operators. The states with $e_{\boldsymbol{I}} = 1$ on every vertex are the string-net states that satisfy the branching rule. Those states are called closed string-net states. The different closed-string-net states are labeled by $b_{\boldsymbol{p}}$. However, the labeling is not one-to-one. The number of the closed-string-net states labeled $b_{\boldsymbol{p}}$ is F_n where n is the number $b_{\boldsymbol{p}}$'s that are equal to 0.

We would like to point out that the hexagon with $b_{\boldsymbol{p}} = 0$ correspond to a quasiparticle excitation at the hexagon $\boldsymbol{p}$. Thus, the state that has n hexagons with $b_{\boldsymbol{p}} = 0$ correspond to a state with n quasiparticle excitations. The above result implies that even when we fixed the locations of the quasiparticles, the corresponding states still have degeneracy. We would like to stress that the degeneracy is topological. No perturbation can lift the degeneracy when the quasiparticles are far apart.

On a periodic Kagome lattice, the ground states of the $N = 1$ string-net model have a four-fold topological degeneracy and contain a non-trivial topological order. However, the topological order in the ground states are very different from those in the $N = 1$ closed-string model discussed in the last subsection. The low energy effective theory of the $N = 1$ string-net models are described by the truncated $SU_3(2) \times SU_3(2)$ non-Abelian Chern-Simons theory and the quasiparticles carry non-Abelian statistics. The topological degeneracy of the n-quasiparticle state makes the non-Abelian statistics possible.

11.11 Long string operators

11.11.1 Quasiparticles and invisible closed string operators

As we have seen in the last two sections that all the eigenstates of the string-net model H_{strnet} (11.8.2) are labeled by the common eigenvalues of $E_{\boldsymbol{I}}$: $e_{\boldsymbol{I}} = 0, 1$ and $B_{\boldsymbol{p}}$: $b_{\boldsymbol{p}} = b_0, b_1, ...$ The ground state corresponds to $e_{\boldsymbol{I}} = 1$ and $b_{\boldsymbol{p}} = b_0$ (b_0 is the maximum eigenvalue of $B_{\boldsymbol{p}}$). Changing a few e_{vI} from 1 to 0 and a few $b_{\boldsymbol{p}}$ from b_0 to $b_1, b_2, ...$ correspond to creating a few quasiparticles. Those quasiparticles have finite energies and short-range interactions. In this section, we are going to discuss the physical properties of those quasiparticle excitations. First we would like to find the creation operators that create those quasiparticles.

This turns out to be a difficult task. One reason is that those quasiparticles cannot be created by local operators in our "spin" model H_{strnet}. For example, in the $N = 1$ closed-string model discussed in the last section, the quasiparticle created by changing e_{vI} from 1 to 0 correspond to an end of open string. There is no way to create a single end of string alone. The ends of string can only be created by a string operator and their can only be created in pairs. (A string operator, by definition, is a operator formed by the product of local operators on a curve.)

Clearly, a string operator will create an excitation above the ground state. In general, such a excitation behave like an extended object (say with an energy proportional to its length). So the string operators that

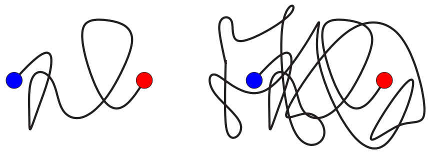


Figure 11.15: (a) An open string with two ends. (b) The open string is unobservable in the background of string-net condensed state. Thus the ends of open strings behave like independent particles.

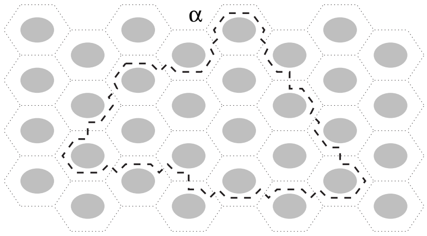


Figure 11.16: The $B^s_{\boldsymbol{p}}$ operator creates a small loop of type-s string. The long string loop is created by the string operator.

create our quasiparticles must be very special. They must create excitations whose energy do not depend on the length of string. Further more, different strings that share the same ends must create exactly the same excitation. That is the string itself (apart from its ends) is unobservable. Is this possible? The answer is yes, but it can only happen in the string condensed state (see Fig. 11.15). The invisibility of a string means that the corresponding closed string operator $W(C_{\text{close}})$ does not create any thing from the ground state. Mathematically, this means that the ground state is an eigenstate of the closed string operator

$$W(C_{\text{close}})|\text{ground}\,\rangle = \lambda|\text{ground}\,\rangle.$$

We may rescale the operator $W(C_{\text{close}})$ to make the eigenvalue to be $\lambda = 1$: In this way, we obtain $\langle\text{ground}\,|W(C_{\text{close}})|\text{ground}\,\rangle = 1$. The above result is very similar to the order parameter characterization of symmetry breaking state: $\langle\text{ground}\,|\phi(\boldsymbol{x})|\text{ground}\,\rangle = \text{constant}$.

We see that there are two pictures for string-net condensation. In the first picture, we view a string-net condensed state as a superposition of different string-nets whose size is as large as the system. This is the picture adapted in this chapter. In the second picture, we may view a string-net condensed state as a common eigenstate of non-trivial closed string operators. The closed strings can have arbitrary shapes and their size can be as large as the system. The second picture is more general, which was used in the last chapter. In the second picture, different string-net condensed states are classified by different algebras of the closed string operators.

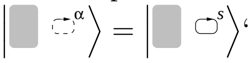
11.11.2 Simple string operators

One way to find invisible string operators is to find string operators that commute with the Hamiltonian except near its end, in other words, to find closed string operator that commute with the Hamiltonian H_{strnet} (11.8.2)

$$[W(C_{\text{close}}), H_{\text{strnet}}] = 0$$

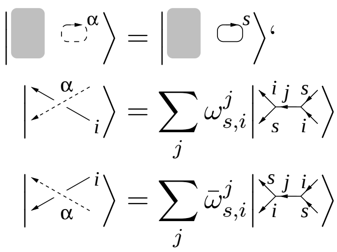
Certainly, a product local identity operators along a loop satisfies the above above condition. (A local identity operator, by definition, is the identity operator that acts within the local Hilbert space on a site.) But obviously, such a string operator is trivial. We need to find non-trivial closed-string operators that commute with H_{strnet}. But what does “non-trivial” mean? One definition of non-trivial closed-string operators is the following. We know that we can always obtain open-string operators from closed-string operator. A non-trivial closed-string operator has a properties that its corresponding open-string operator never commute with the Hamiltonian H_{strnet} no matter how we modify its ends (by multiplying local operators near the ends of the open string).

First let us try to find the string operator that creates quasiparticle corresponding to $e_{\boldsymbol{I}} = 0$. Those quasiparticles are ends of strings. From the graphic representation of the $B^s_{\boldsymbol{p}}$ operator, we see that $B^s_{\boldsymbol{p}}$ creates a small loop of type-s string around a hexagon (see Fig. 11.12). Also $B^s_{\boldsymbol{p}}$ commute with the H_{strnet}. So if we can generalize $B^s_{\boldsymbol{p}}$ to a closed string with any length and shape, the resulting operator, denoted as $W_s(C)$, will be the desired closed-string operator.

Since the $B^s_{\boldsymbol{p}}$ is obtained by simply adding a small loop of string, its natural generalization appear be adding a large loop of string (see Fig. 11.16). Here we would like to point out that the “string” in the string operator and the “string” in the string-net condensed state are quite different things. The “string” in the operator represents a product of operators along the string, while the “string” in the string-net condensed state represents a collection of “flipped” spins. So we will use dash line to represent “string” in the operator. Its the action of the string operator that produces the strings in the state. Or in other wrods, the action of the dash line produces solid line: 

The explicit form of the matrix elements of $B^s_{\boldsymbol{p}}$ are obtained from the local rules (11.4.1) – (11.4.4). Similarly to obtain the matrix elements of $W_s(C)$ that creates a long string, we can start with its graphic representation (see Fig. 11.16) and then use local rules.

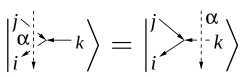
We note that the long string and the small loop created by $B^s_{\boldsymbol{p}}$ have an important difference. The small loop created by $B^s_{\boldsymbol{p}}$ never cross the strings that are already in the state. (The strings in the state always live on the links.) However, a long string in general does cross those strings. So we need additional local rules to handle the crossing. It turns out that the following set of local rules for string operators allow us completely determine the their matrix elements:



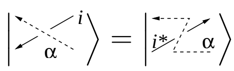
(11.11.1)

where $\omega^j_{s,i}$ and $\bar\omega^j_{s,i}$ are complex numbers. Eqn (11.11.1) in fact defines the close string operator W_s.

Since we are interested in the invisible closed-string operators whose action does depend on the shape and the position of the strings, so we require $\omega^j_{s,i}$ and $\bar\omega^j_{s,i}$ to satisfy



(11.11.2)



(11.11.3)

Using the local roles (11.9.1) and (11.11.1), we can rewrite the above as algebraic equations

$$\bar\omega^m_{s,j} F^{sl^*i}_{kjm^*} \omega^l_{s,i} \frac{v_j v_s}{v_m} = \sum_{n=0}^{N} F^{ji^*k}_{s^*nl^*} \omega^n_{s,k} F^{jl^*n}_{ksm^*}$$

$$\bar\omega^j_{s,i} = \sum_{k=0}^{N} \omega^k_{s,i^*} F^{is^*k}_{i^*sj*} \tag{11.11.4}$$

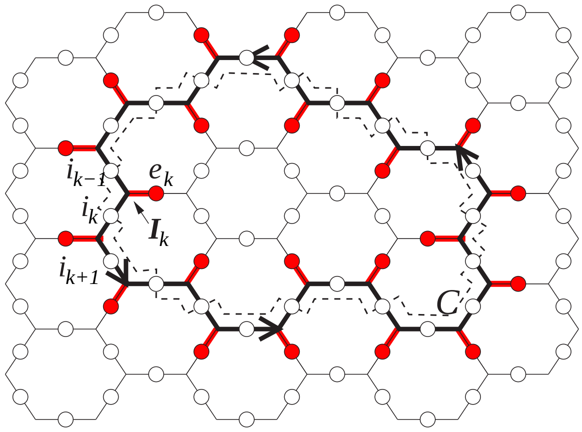


Figure 11.17: The placement of the string operator $W_s(C)$. The think lines are links that form the oriented loop C. The links attached to the loop C are the legs of C. The spin states on the legs are labeled by e_k. The spin states on C are labeled by i_k. The vertices on C are labeled by $\boldsymbol{I}_k$. We note that the string operator cross the link k only when C turns (right, left) or (left, right) at (I_k, I_{k+1}).

Note that s in the above equations is fixed.

After finding the solutions of eqn (11.11.4), we can calculate the matrix elements using the local rules (11.9.1) and (11.11.1). The placement of the closed string operator follows the convention in Fig. 11.17. We find that the string operator $W_s(C)$ only changes the spin states on the path C (see Fig. 11.17). The matrix elements also depend on spins on the legs of C. The matrix element of string operator $W_s(C)$ between an initial spin state $i_1, ...i_N$ and final spin state $i'_1, ...i'_N$ on the path is of the form

$$(W_s(C))^{i'_1 i'_2 ... i'_N}_{i_1 i_2 ... i_N}(e_1 e_2 ... e_N) = \left(\prod_{k=1}^{N} F^s_k\right)\left(\prod_{k=1}^{N} \omega_{s,k}\right) \tag{11.11.5}$$

where $e_1, ..., e_N$ are the spin states of the N "legs" of C (see Fig. 11.17) and

$$F^s_k = \begin{cases} F^{e_k i^*_k i_{k-1}}_{s^* i'_{k-1} i'^*_k}, & \text{if } C \text{ turns left at } \boldsymbol{I_k} \\ F^{e_k i^*_{k-1} i_k}_{s i'_k i'^*_{k-1}}, & \text{if } C \text{ turns right at } \boldsymbol{I_k} \end{cases} \tag{11.11.6}$$

$$\omega_{s,k} = \begin{cases} \frac{v_{i_k} v_s}{v_{i'_k}} \omega^{i'_k}_{s,i_k}, & \text{if } C \text{ turns right, left at } \boldsymbol{I_k}, \boldsymbol{I_{k+1}} \\ \frac{v_{i_k} v_s}{v_{i'_k}} \bar\omega^{i'_k}_{s,i_k}, & \text{if } C \text{ turns left, right at } \boldsymbol{I_k}, \boldsymbol{I_{k+1}} \\ 1, & \text{otherwise} \end{cases} \tag{11.11.7}$$

Because of (11.11.2), one can show that the above long string operator commute with $B^s_{\boldsymbol{p}}$ and $E_{\boldsymbol{I}}$, and hence H_{strnet}, using the graphic representation.

We can easily obtain open string operator from the above closed string operator. Such a open string operator will create a pair of quasiparticles that correspond to ends of type-s string.

However, the equations (11.11.4) have more then one solutions! The different solutions give rise to different long string operators that all create a loop of type-s string when acting on a state without strings. However, the action of those different long string operators on a state with strings are different.

To understand the meaning of those different solutions, let us consider the case of Abelian gauge theory. The solutions to (11.11.4) can be divided into three classes. The first class is given by $\omega^j_{s,i} \frac{v_i v_s}{v_j} = \bar\omega^j_{s,i} \frac{v_i v_s}{v_j} = 1$ for $s \neq 0$. These string operators create type-s strings that correspond to electric flux lines in the gauge theory. The corresponding open string operators create a pair of electric charges. The second class of solutions is given by $\omega^j_{0,i}\frac{v_i}{v_j} = (\bar{\omega}^j_{0,i}\frac{v_iv_s}{v_j})^* \neq 1$ for $s = 0$. These string operators create no strings, but they modify the string-net wave function. Those operators are similar to the vortex creation operators in the superfluid. The associated open string operators create a pair of quasiparticles that correspond to changing $b_{\boldsymbol{p}}$ for b_0 to some other values. Those quasiparticles correspond to the magnetic-flux excitation in the gauge theory. The third class has $s \neq 0$ and $\omega^j_{s,i}\frac{v_iv_s}{v_j} = (\bar{\omega}^j_{s,i}\frac{v_iv_s}{v_j})^* \neq 1$. The corresponding open string operators create quasiparticles which correspond to electric-charge/magnetic-flux bound states. This accounts for all the quasiparticles in $(2+1)D$ Abelian gauge theory. Therefore, all the string operators are simple in this case.

Bibliography

Ardonne, E., P. Fendley, and E. Fradkin, 2004, *Topological Order and Conformal Quantum Critical Points*, Annals Phys. **310**, 493.

Arovas, D., J. R. Schrieffer, and F. Wilczek, 1984, *Fractional Statistics and the Quantum Hall Effect*, Phys. Rev. Lett. **53**, 722.

Balents, L., M. P. A. Fisher, and S. M. Girvin, 2002, *Fractionalization in an Easy-axis Kagome Antiferromagnet*, Phys. Rev. B **65**, 224412.

Banks, T., R. Myerson, and J. B. Kogut, 1977, *Phase Transitions In Abelian Lattice Gauge Theories*, Nucl. Phys. B **129**, 493.

Blok, B., and X.-G. Wen, 1990, *Effective theories of Fractional Quantum Hall Effect: Hierarchical Construction*, Phys. Rev. B **42**, 8145.

Foerster, D., H. B. Nielsen, and M. Ninomiya, 1980, *Dynamical stability of local gauge symmetry – Creation of light from chaos*, Phys. Lett. B **94**, 135.

Freedman, M., M. Larsen, and Z. Wang, 2002, *A modular functor which is universal for quantum computation*, Commun. Math. Phys. **227**, 605.

Freedman, M., C. Nayak, and K. Shtengel, 2003*a*, *Non-Abelian topological phases in an extended Hubbard model*, cond-mat , 0309120.

Freedman, M., C. Nayak, K. Shtengel, K. Walker, and Z. Wang, 2003*b*, *A Class of P, T-Invariant Topological Phases of Interacting Electrons*, cond-mat , 0307511.

Fröhlich, J., and T. Kerler, 1991, Nucl. Phys. B **354**, 369.

Ginzburg, V. L., and L. D. Landau, 1950, *On the theory of superconductivity*, Zh. Ekaper. Teoret. Fiz. **20**, 1064.

Hansson, T. H., V. Oganesyan, and S. L. Sondhi, 2004, *Superconductors are topologically ordered*, cond-mat/0404327 .

Ioffe, L. B., M. V. Feigel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, 2002, *Topologically protected quantum bits from Josephson junction arrays*, Nature **415**, 503.

Kalmeyer, V., and R. B. Laughlin, 1987, *Equivalence of the resonating-valence-bond and fractional quantum Hall states*, Phys. Rev. Lett. **59**, 2095.

Kassel, C., 1995, *Quantum Groups* (Springer-Verlag, New York).

Kitaev, A. Y., 2003, *Fault-tolerant quantum computation by anyons*, Ann. Phys. (N.Y.) **303**, 2.

Kogut, J., and L. Susskind, 1975, *Hamiltonian formulation of Wilson's lattice gauge theories*, Phys. Rev. D **11**, 395.

Kogut, J. B., 1979, *An introduction of Lattice gauge theory and spin systems*, Rev. Mod. Phys. **51**, 659.

Landau, L. D., 1937, Phys. Z. Sowjetunion **11**, 26.

Levin, M., and X.-G. Wen, 2003, *Fermions, strings, and gauge fields in lattice spin models*, Phys. Rev. B **67**, 245316.

Moessner, R., and S. L. Sondhi, 2001, *Resonating Valence Bond Phase in the Triangular Lattice Quantum Dimer Model*, Phys. Rev. Lett. **86**, 1881.

Moore, G., and N. Seiberg, 1989, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123**, 177.

Motrunich, O. I., and T. Senthil, 2002, *Exotic order in simple models of bosonic systems*, Phys. Rev. Lett. **89**, 277004.

Read, N., 1990, *Excitation structure of the hierarchy scheme in the fractional quantum Hall effect*, Phys. Rev. Lett. **65**, 1502.

Read, N., and B. Chakraborty, 1989, *Statistics of the excitations of the resonating-valence-bond state*, Phys. Rev. B **40**, 7133.

Read, N., and S. Sachdev, 1991, *Large-N expansion for frustrated quantum antiferromagnets*, Phys. Rev. Lett. **66**, 1773.

Rokhsar, D. S., and S. A. Kivelson, 1988, *Superconductivity and the Quantum Hard-Core Dimer Gas*, Phys. Rev. Lett. **61**, 2376.

Sachdev, S., and K. Park, 2002, *Ground states of quantum antiferromagnets in two dimensions*, Annals of Physics (N.Y.) **298**, 58.

Senthil, T., and M. P. A. Fisher, 2000, *Z_2 gauge theory of electron fractionalization in strongly correlated systems*, Phys. Rev. B **62**, 7850.

Turaev, V. G., 1994, *Quantum invariants of knots and 3-manifolds* (W. de Gruyter, Berlin-New York).

Wegner, F., 1971, J. Math. Phys. **12**, 2259.

Wen, X.-G., 1990, *Topological Orders in Rigid States*, Int. J. Mod. Phys. B **4**, 239.

Wen, X.-G., 1991*a*, *Mean Field Theory of Spin Liquid States with Finite Energy Gaps*, Phys. Rev. B **44**, 2664.

Wen, X.-G., 1991*b*, *Topological Orders and Chern-Simons Theory in strongly correlated quantum liquid*, Int. J. Mod. Phys. B **5**, 1641.

Wen, X.-G., 1995, *Topological Orders and Edge Excitations in FQH States*, Advances in Physics **44**, 405.

Wen, X.-G., 2002*a*, *Origin of Gauge Bosons from Strong Quantum Correlations (Origin of Light)*, Phys. Rev. Lett. **88**, 11602.

Wen, X.-G., 2002*b*, *Quantum Orders and Symmetric Spin Liquids (extended version at http://arXiv.org/abs/cond-mat/0107071)*, Phys. Rev. B **65**, 165113.

Wen, X.-G., 2003*a*, *Artificial light and quantum order in systems of screened dipoles*, Phys. Rev. B **68**, 115413.

Wen, X.-G., 2003*b*, *Quantum order from string condensations and origin of light and massless fermions*, Phys. Rev. D **68**, 065003.

Wen, X.-G., and Q. Niu, 1990, *Ground State Degeneracy of the FQH States in Presence of Random Potentials and on High Genus Riemann Surfaces*, Phys. Rev. B **41**, 9377.

Wen, X.-G., F. Wilczek, and A. Zee, 1989, *Chiral Spin States and Superconductivity*, Phys. Rev. B **39**, 11413.

Wen, X.-G., and A. Zee, 2002, *Gapless Fermions and Quantum Order*, Phys. Rev. B **66**, 235110.

Witten, E., 1989*a*, *Gauge theories and integrable lattice models*, Nuclear Physics B **322**, 629.

Witten, E., 1989*b*, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121**, 351.

Witten, E., 1990, *Gauge theories, vertex models, and quantum groups*, Nuclear Physics B **330**, 285.