Spin-singlet superfluid state for spin-1 bosons with fractional spin and statistics

W. Vincent Liu[1, 2] and Xiao-Gang Wen[2]

[1] *Center for Theoretical Physics, Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

[2] *Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

We study a strongly correlated spin-1 Bose gas in 2D space by using the projective construction. A spin-singlet superfluid state is constructed and proposed as a candidate competing with the conventional polar condensate when interaction is antiferromagnetic. This novel state has a nontrivial topological order whose low energy excitations carry fractional spin, charge, and statistics. The spin excitations become gapless only at the edge and are described by level-1 $\mathrm{SU}(2)_{\mathrm{spin}}$ Kac-Moody algebra. The edge state is identical to the edge state of the chiral spin liquid or the right moving sector of spin-1/2 chain.

So far most studies of ultra cold alkali atomic gases have been focused on Bose-Einstein condensation (BEC) in weakly interacting dilute limit (for review, see for instance [1]). By contrast, a strongly correlated Bose gas may lead us to interesting novel phenomena. While it is unclear whether one can achieve a *stable* strongly correlated gas in 3D by simply cranking up the scattering length without collapsing the system, a 2D gas with even relatively weak interaction can be strongly correlated in nature. One can see this by simple renormalization group argument. When the characteristic momentum scales down $\mathbf{k} \to \kappa\mathbf{k}$, a generic four-boson interaction operator scales κ^{d-2}, so the interaction is irrelevant in $d = 3$, marginal in $d = 2$, and relevant in $d = 1$ when $\kappa \to 0$. A thorough RG analysis of the 2D dilute Bose gas [2] showed that the interaction is marginally irrelevant only in a dilute limit specified by $\ln\ln(1/na^2) \gg 1$, where n is the particle density and a can be thought of the interaction range or the scattering length. The double logarithm imposes a much stronger condition of the validity of the dilute limit in 2D than the familiar $na^3 \ll 1$ in 3D. In fact, long before Ref.[2], the change of the diluteness condition was implied in Popov's [3] diagrammatic dilute gas expansion. For generic BEC system such as the disk condensate of ^{23}Na of Ref. [4] or a microelectronic chip of condensed ^{87}Rb atoms [5], $\ln\ln(1/na^2)$ is of $O(1)$. These systems are, at least, not weakly correlated. Now the question is that what is the ground state for those not-weakly correlated 2D boson gas? In this letter, we shall show that a new class of 2D superfluid can emerge due to the strong correlation.

Let us consider a strongly correlated gas of spin-1 bosons ϕ_m $(m = 0, \pm 1)$ in a 2D homogeneous space with generic two-body interactions $H^{\mathrm{int}} = \int d^2\mathbf{r} d^2\mathbf{r}' \sum_{F=0}^{2} \mathscr{H}_F^{\mathrm{int}}$ with [6–8]

$$\mathscr{H}_F^{\mathrm{int}} = \tfrac{1}{2} V_F(\mathbf{r}-\mathbf{r}') C^{FF_z}_{mm'} C^{FF_z}_{n'n} \phi^\dagger_m(\mathbf{r}) \phi^\dagger_{m'}(\mathbf{r}') \phi_{n'}(\mathbf{r}') \phi_n(\mathbf{r}) \tag{1}$$

where $C^{FF_z}_{m_1m_2} = \langle FF_z | 1m_1 1m_2\rangle$ are the Clebsch-Gordan coefficient for total hyperfine spin $\mathbf{F} = \mathbf{1} + \mathbf{1}$. In the alkali atomic gases, the interactions for all three spin channels V_F, $F = 0, 1, 2$ are commonly short-ranged. The Hamiltonian has a phase and spin rotation symmetry: $\mathrm{U}(1)_{\mathrm{charge}} \times \mathrm{SU}(2)_{\mathrm{spin}}$. The Hamiltonian first derived by Ho [6] for spinor condensates, $\mathscr{H}^{\mathrm{int}} = \frac{1}{2}(c_0 \hat{n}^2 + c_2 \hat{\mathbf{F}}^2)$ where $\hat{n}$ and $\hat{\mathbf{F}}$ are the density and spin operators, corresponds to a special form of (1) with $V_0(\mathbf{r}-\mathbf{r}') = (c_0 - 2c_2)\delta(\mathbf{r}-\mathbf{r}')$, $V_1(\mathbf{r}-\mathbf{r}') = (c_0 - c_2)\delta(\mathbf{r}-\mathbf{r}')$, and $V_2(\mathbf{r}-\mathbf{r}') = (c_0 + c_2)\delta(\mathbf{r}-\mathbf{r}')$. Recent mean-field studies of such a system [6, 7, 9] suggest that the spinor condensate leads to many quantum phenomena that are absent in the scalar condensate. The first attempt to explore the spin correlation we believe was made by Law *et al.* [8], who studied an extreme limit where the spin degree freedom has no spatial dependence and found a total spin singlet state (instead of a polar condensate) for antiferromagnetic interactions. Later on, Ho and Yip [6] showed that the spin-singlet state of Ref. [8] is however a fragmented condensate with fragile stability. Putting the spinor BEC in an optical lattice, Demler and Zhou speculated possible spin singlet pairs and fractionalized spin excitations for arbitrary dimensions based on a Z_2 gauge effective theory [10].

For an antiferromagnetic interaction ($V_2 > V_0$ or correspondingly $c_2 > 0$ in Ref. [6]), mean field theory suggests that the polar condensate ($\langle \phi_{m=0} \rangle \neq 0$), which directly condenses bosons into the $m = 0$ channel, is likely favored for it guarantees $\langle \mathbf{F} \rangle = 0$ [6]. However, all spins in such a state line up in the same direction. So it does not optimally minimize the spin part of interaction energy. We find $\langle \mathscr{H}^{\mathrm{int}}_{F=0} \rangle_{\mathrm{polar}} = \frac{1}{6} V_0(\delta\mathbf{r}) n_0^2$, $\langle \mathscr{H}^{\mathrm{int}}_{F=1} \rangle_{\mathrm{polar}} = 0$, and $\langle \mathscr{H}^{\mathrm{int}}_{F=2} \rangle_{\mathrm{polar}} = \frac{1}{3} V_2(\delta\mathbf{r}) n_0^2$ with $\delta\mathbf{r} \equiv \mathbf{r}-\mathbf{r}'$. Here, n_0 is the density of condensed bosons and quantum fluctuations are not included. One sees that the $\langle \mathscr{H}^{\mathrm{int}}_{F=2} \rangle_{\mathrm{polar}}$ is even greater than that for $F = 0$ if $V_2 = V_0$. This motivates us to search for a spin singlet condensate that minimizes the spin interaction energy better. We have found such a (p-wave) state which has: $\langle \mathscr{H}^{\mathrm{int}}_{F=0} \rangle_{\mathrm{singlet}} = \frac{1}{6} V_0(\delta\mathbf{r}) n^2 + O(|\delta\mathbf{r}|^4)$, $\langle \mathscr{H}^{\mathrm{int}}_{F=1} \rangle_{\mathrm{singlet}} \sim -(\#)\frac{\pi}{3} V_1(\delta\mathbf{r}) n^4 |\delta\mathbf{r}|^4$, and $\langle \mathscr{H}^{\mathrm{int}}_{F=2} \rangle_{\mathrm{singlet}} = \frac{5\pi^2}{9} V_2(\delta\mathbf{r}) n^4 |\delta\mathbf{r}|^4$, where n is the total density of the bosons. Notice that the last two terms vanish when $\delta\mathbf{r} \to 0$. (Detailed calculations shall be given elsewhere.) Thus, the spin-singlet superfluid has a lower energy when V_2 is positively large and short ranged.

To study the above mentioned spin-singlet superfluid, we begin by constructing its ground state wave function. In Ref. [11], a projective construction is introduced to construct spin-singlet quantum Hall states of spin-1 bosons. The same construction can be used to obtain the spin-singlet superfluid. In the projective construction, we introduce two spin-$\frac{1}{2}$ fermions, $\psi_{a\alpha}$, each with two (physical) spins $\alpha =\uparrow,\downarrow$ and two fictitious "colors" labelled by $a, b, \ldots$. The color is necessary to furnish a minimal spin-1 bosonic representation at every space-time point. The boson can then be represented by

$$\phi_m(\mathbf{r}) = \mathscr{N}_\psi \psi_{a\alpha}(\mathbf{r})\psi_{b\beta}(\mathbf{r})\epsilon_{ab}\mathscr{C}^m_{\alpha\beta}\,, \tag{2}$$

where $\mathscr{N}_\psi$ is a normalization constant with a spatial dimension and $\mathscr{C}^m_{\alpha\beta} = \langle 1m|\frac{1}{2}\alpha\frac{1}{2}\beta\rangle$ are the Clebsch-Gordan coefficient with $m = 0, \pm 1$. Note that the color degree of freedom is non-physical. All physical states or operators are required invariant under a *local* $\mathrm{SU(2)}_{\mathrm{color}}$ transformation: $\psi_{a\alpha}(\mathbf{r}) \to W(\mathbf{r})_{ab}\psi_{b\alpha}(\mathbf{r})$, where $W(\mathbf{r})$ is an $\mathrm{SU(2)}_{\mathrm{color}}$ matrix. One can quickly check that the boson operator ϕ_m defined in (2) is indeed a color singlet, perfectly invariant under above transformation. Eq. (2) allows us to construct physical boson many-body wave function $|\Psi^b\rangle$ from the unphysical fermion many-body wave function$|\Psi^f\rangle$ by projecting it into the "color" singlet sector. In a mathematical equation, that means

$$\Psi^b(\mathbf{r}_1, m_1; \mathbf{r}_2, m_2; ...) = \langle 0|\textstyle\prod_i \phi_{m_i}(\mathbf{r}_i)|\Psi^f\rangle\,. \tag{3}$$

The relationship (3) suggests that the effective theory of our spin-1 boson system can either be described in terms of the boson operator ϕ_m, or equivalently, in terms of the fermion operator $\psi_{a\alpha}$. Unfortunately, for a strongly correlated 2D system, there is no known rigorous way to derive the low energy effective theory from a microscopic Hamiltonian like (1). One usually first writes down a most natural effective theory on symmetry grounds, checks its stability against interactions in low energy limit, and finally compares it with experiments. In this spirit, the low energy effective theory for the state $\Psi^b(\mathbf{r}_1, m_1; \mathbf{r}_2, m_2; ...)$ is described, in the fermion description, by a theory of independent fermions coupled to color SU(2) gauge fields. The gauge field is denoted as $A_\mu \equiv \frac{1}{2}\tau^l A^l_\mu$, $l = 1, 2, 3$, where τ^l are the Pauli matrices generating the $\mathrm{SU(2)}_{\mathrm{color}}$ algebra. The gauge fields are introduced to project out the unphysical colored excitations.[11] The effective theory is then

$$\begin{aligned}\mathscr{L} \;&=\; i\psi^\dagger_{a\alpha}(D_0)_{ab}\psi_{b\alpha} + \frac{1}{2M}\psi^\dagger_{a\alpha}(\boldsymbol{D}\cdot\boldsymbol{D})_{ab}\psi_{b\alpha} \\ &\quad +\text{all symmetry allowed interactions}\,,\end{aligned} \tag{4}$$

where $(D_\mu)_{ab} \equiv \delta_{ab}\partial_\mu - i(A_\mu)_{ab}$ (for notation, see [12]) are the covariant derivatives. From the fermion effective theory, we can study various fermion states, which, after the projection (3), lead to various physical boson states.

To see which fermion states are likely to appear as the ground state, we need to consider the interactions between the fermions. Interactions can be either originated from the boson-boson interactions or dynamically generated by gauge interactions. As an example of non-Abelian gauge theory, the study of QCD [13] shows that the Yang-Mills gauge fluctuations can generate a strong attractive interaction between quarks due to the instanton effect, which leads to quark confinement. In our case, the gauge fields are used to mediate a strong attractive interaction between color-opposite fermions, since by definition (2) the gauge interaction is supposed to bind two fermions locally into a colorless boson. Therefore, the strong SU(2) gauge interaction naturally leads to color-singlet Cooper pairing.

Let us consider two most natural, translationally and rotationally invariant color-singlet parings:

$$\begin{aligned}&\langle\psi_{a\alpha}(\mathbf{r})\psi_{b\beta}(0)\rangle \\ &= \begin{cases} \epsilon_{ab}\,\mathscr{C}^{\bar m}_{\alpha\beta}\,R_s(|\mathbf{r}|)\,, & (s,\ \text{spin triplet}) \\ \epsilon_{ab}\,\epsilon_{\alpha\beta}\,R_p(|\mathbf{r}|)\,(x+iy)\,, & (p_{x+iy},\ \text{spin singlet})\end{cases}\end{aligned} \tag{5}$$

where $\bar m$ can be 0 or 1. Both s- and p_{x+iy}-wave states produce a full gap on the Fermi surface. $R_{s,p}(|\mathbf{r}|)$ are complex functions of $|\mathbf{r}|$, generically expected to monotonically fall off exponentially at large distance.

One may wonder why we have not included a p_x-wave paring in above: $\langle\psi_{a\alpha}(\mathbf{r})\psi_{b\beta}(0)\rangle \sim \epsilon_{ab}\,\epsilon_{\alpha\beta}\,R_p(|\mathbf{r}|)\,x$. The reason is twofold. First, unlike those pairings in (5), it has gapless fermions at two nodes on the Fermi surface and the gauge fields are gapless. Therefore, the p_x-wave state is unstable against the strongly relevant gauge interaction. Secondly, as known in the study of liquid ^{3}He [14], a simple weak-coupling theory favors the most uniformly distributed gap function possible, which means the p_{x+iy} is energetically favored over the p_x, if the system intends to pair in p-wave channel anyway.

a. s-wave pairing: conventional BEC phases. Conventional BEC phases are easily recovered in our method through the s-wave paring channel in the limit of strong confinement, where $R_s(|\mathbf{r}|)$ becomes a delta function $\sim \delta(\mathbf{r})$.

Polar condensate. This is a special kind of spin nematic state. Fermion confinement occurs in spin-1, $m = 0$ channel. Here, the pairing in Eq.(5) reduces to $\langle\psi_{a\alpha}(\mathbf{r})\psi_{b\beta}(\mathbf{r})\rangle = \sqrt{\rho}e^{i\varphi}\epsilon_{ab}\mathscr{C}^{m=0}_{\alpha\beta}$ where one may think of ρ related to the condensed boson density. The resulting state is nothing but the so-called polar condensate with $\langle\phi_{m=0}\rangle \neq 0$. This state breaks both $\mathrm{U(1)}_{\mathrm{charge}}$and $\mathrm{SU(2)}_{\mathrm{spin}}$ invariance, and was considered to be favored if the spin interaction is antiferromagnetic ($V_2 > V_0$) [6].

Ferromagnetic condensate. This case is the same as the polar condensate except that the fermions are confined into the $m = 1$ (or equivalently -1) channel. The order parameter becomes $\langle\psi_{a\alpha}(\mathbf{r})\psi_{b\beta}(\mathbf{r})\rangle = \sqrt{\rho}e^{i\varphi}\epsilon_{ab}\mathscr{C}^{m=1}_{\alpha\beta}$ which corresponds to the ferromagnetic condensate. Like the polar condensate, it breaks both $\mathrm{U(1)}_{\mathrm{charge}}$and $\mathrm{SU(2)}_{\mathrm{spin}}$ invariance. This state is presumably favored if the spin interaction is ferromagnetic, $V_2 < V_0$.

b. p_{x+iy}-state: topological superfluid. As we have argued near the beginning of the paper, the spin-singlet p_{x+iy}-wave pairing naturally competes with the (spin-triplet) polar condensate in optimally minimizing an antiferromagnetic interaction represented by a large positive V_2. Assuming the state to exist, it becomes a standard routine to construct its low energy effective theory, simply based on the broken (physical) symmetries without relying on microscopic details. We find the effective Lagrangian is given by

$$\int_{\mathbf{r}} \left\{ \psi^\dagger_{a\alpha} i(D_0)_{ab}\psi_{b\alpha} + \frac{1}{2M}\psi^\dagger_{a\alpha}\boldsymbol{D}^2_{ab}\psi_{b\alpha} \right\} + \qquad (6)$$
$$\int_{\mathbf{r}\mathbf{r}'} \left[\psi^\dagger_{a\alpha}(\mathbf{r})\Delta(\mathbf{r},\mathbf{r}')\epsilon_{\alpha\beta}[e^{i\int_{\mathbf{r}'}^{\mathbf{r}}\boldsymbol{A}\cdot\boldsymbol{dl}} i\tau^2]_{ab}\psi^\dagger_{b\beta}(\mathbf{r}) + \text{h.c.}\right] ,$$

where $\Delta(\mathbf{r},\mathbf{r}')$ is the p-wave pairing wavefunction. In the momentum space it becomes $\Delta(\mathbf{k}) \equiv \frac{\Delta_0}{2}(k_x + ik_y)$. The theory is evidently invariant under the global $\mathrm{SU}(2)_{\mathrm{spin}}$ and local $\mathrm{SU}(2)_{\mathrm{color}}$ transformations. However, since the p_{x+iy}-wave condensate (see Eq. (5)) spontaneously breaks the charge U(1), the phase fluctuations of Δ gives rise to the usual (gapless) superfluid mode. The charge sector therefore is rather conventional. In contrast, we shall show that the spin sector has excitations with fractional spin and statistics.

Let us focus on the spin sector of the effective theory (6), by regarding Δ as a fixed non-dynamical field. To see the symmetry more clearly, we introduce four 2-component spinors, each uniting a pair of particle and hole operators:

$$\eta_{a\alpha} \equiv \begin{pmatrix} \psi_{a\alpha} \\ \epsilon_{ab}\epsilon_{\alpha\beta}\psi^\dagger_{b\beta} \end{pmatrix} , \qquad (a,b = 1,2;\ \alpha,\beta = \uparrow,\downarrow) . \quad (7)$$

In terms of η's, the effective action in momentum space becomes

$$I_{\text{eff}} = -\tfrac{1}{2}\int d^3k \eta^\dagger_{a\alpha}(k)[\mathscr{D}(k^\mu - A^\mu)]_{a\alpha;b\beta}\eta_{b\beta}(k) , \quad (8)$$
$$\mathscr{D}(k) = \hat\tau^0 \otimes \tau^0 \otimes [-\omega + \vec h(\mathbf{k})\cdot\vec\sigma] ,\ k^\mu \equiv (\omega, \mathbf{k}) , \quad (9)$$
$$\vec h(\mathbf{k}) \equiv \left(\Delta_0 k_y\, ,\ \Delta_0 k_x\, ,\ \mathbf{k}^2/2M - \mu\right) . \quad (10)$$

The $\mathscr{D}(k)$ is better viewed as a 8×8 matrix in an orthogonal basis $\{\hat\tau^0 = \mathbb{1}, \hat\tau^l\}_{\text{spin}} \otimes \{\tau^0 = \mathbb{1}, \tau^l\}_{\text{color}} \otimes \{\sigma^0 = \mathbb{1}, \sigma^l\}$ $(l = 1,2,3)$ with the σ's acting on the "Nambu" spinor space (7). A careful reader may recall $A_\mu = \frac{1}{2}\tau^l a^l_\mu$.

The spectrum of fermionic excitations is determined by the pole of the inverse of $\mathscr{D}(k)$ at $A_\mu = 0$, $E(\mathbf{k}) = \sqrt{(\mathbf{k}^2/2M - \mu)^2 + \Delta_0^2}$. The fermions are fully gapped, so we integrate out them to find the effective action of gauge fields. To do so, we have chosen a background field gauge in (8) such that $A_\mu = Const$ but non-commuting, $[A_\mu, A_\nu] \neq 0$. The approach we adopted here is standard in non-Abelian gauge field theory. Since the action (8) is quadratic in fermion fields, the gauge effective Lagrangian is exactly given by

$$\mathscr{L}_{\text{eff}}[A] = -\tfrac{i}{2(2\pi)^3}\int d^3k \mathrm{Tr}\ln\mathscr{D}(k - A) \qquad (11)$$

where the 'Tr' is over the internal space of $\vec{\hat\tau}\otimes\vec\tau\otimes\vec\sigma$. With the $\mathscr{D}$ matrix given in (9), a straightforward calculation gives

$$\mathscr{L}_{\text{eff}}[A] = \tfrac{Q}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{tr}\{A_\mu\partial_\nu A_\rho - \tfrac{2i}{3}A_\mu A_\nu A_\rho\} + \cdots , \quad (12)$$

where $Q = 1$ is a topological charge (winding number) and the '$\cdots$' stands for higher derivatives, including a Maxwell term. Now the 'tr' in (12) is over only the color gauge indices. The effective theory of gauge bosons is thus a level 1 (non-Abelian) SU(2) Chern-Simons theory. All gauge excitations gain a dynamically generated topological mass (i.e., gapped). The gauge interaction becomes short ranged. So the effective theory we found in (6) is stable.

The winding number Q actually can be calculated in a simple way. One first variationally differentiates both sides of (11) three times with respect to the gauge field $\frac{\delta^3\cdots}{\delta A^l_\mu \delta A^m_\nu \delta A^n_\rho}$, sends all $A^l_\mu \to 0$, and then contracts it with $\epsilon_{\mu\nu\rho}\epsilon^{lmn}$. One finds Q is given by a topological invariant,

$$Q = \tfrac{1}{8\pi}\epsilon_{ij}\int d^2\mathbf{k}\,\hat h\cdot\left(\frac{\partial\hat h}{\partial k_i}\times\frac{\partial\hat h}{\partial k_j}\right) , \quad \hat h \equiv \vec h/|\vec h| . \quad (13)$$

Inserting the $\vec h(\mathbf{k})$ functions defined in (10) gives $Q = 1$.

c. Physical properties of the p_{x+iy}-state After obtaining the low energy effective theory, we are ready to study the measurable properties of the state. First the p_{x+iy}-state is a superfluid which does not break spin rotation symmetry. The only gapless excitation is the superfluid mode described by the phase of Δ. All spin excitations have a finite energy gap. Due to the Chern-Simons term, the $SU(2)$ gauge field is not confining. Thus the excitations described by ψ (or η) have a finite energy gap (instead of infinite energy gap). Those excitations carry spin-1/2 and one half of boson charge! They also have a semion statistics (*i.e.*, the statistical angle is $\theta = \pi/2$, right between boson and fermion), as implied by the level-1 $SU(2)$ Chern-Simons terms. The spin rotation symmetry implies that $\langle\phi_m\rangle = 0$ for all m. However, two-boson and three-boson operators both can have finite expectation values. In short distance, $\langle\phi_m(z_1)\phi_{m'}(z_2)\rangle \sim -4\sqrt3(z_1 - z_2)^2 C^{00}_{mm'}$ and $\langle\phi_{m_1}(z_1)\phi_{m_2}(z_2)\phi_{m_3}(z_3)\rangle \sim 8\sqrt2(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)\epsilon_{m_1m_2m_3}$ where $z \equiv x + iy$. The spin-singlet superfluid has an unusual off-diagonal long range order. We see that the minimum vortex has one unit of quantized vorticity. Our p_{x+iy} spin-singlet superfluid breaks the parity and time reversal symmetry. The total angular momentum of the ground state is $\hbar$ per boson. (Note such a total angular momentum is equal to the total angular momentum of usual boson superfluid with one vortex at its center.) Thus, spinning the bosons may help to create the p_{x+iy} spin-singlet superfluid.

d. Edge excitations In the η bases, the mean-field fermion Hamiltonian (described by (8) with A_μ being set to zero) contains four identical 2×2 blocks: $H = \vec h(\mathbf{k})\cdot\vec\sigma$. In each block, the function $\vec h(\mathbf{k})$ defines a mapping from the $\mathbf{k}$-space to S^2 with a winding number 1. This non-trivial winding number leads to a unit Hall conductance.[15, 16] Since each block contributes a unit Hall conductance, it leads to an edge state similar to the one from $\nu = 1$ quantum Hall state,[17] as required by gauge invariance.[18–20] Such an edge state can be described by one chiral fermion field $\lambda_{a\alpha}$ (one for each block). Therefore, if we ignore the $SU(2)$ gauge fluctuations, the mean-field gapless edge excitations of our p_{x+iy}-state are described by the following effective theory: $\lambda^\dagger_{a\alpha} i(\partial_t - v\partial_x)\lambda_{a\alpha}$. Only two of the four λ's are independent because $\eta_{a\alpha}$ (see (7)) are Majorana fermions and each $\lambda_{a\alpha}$ is obtained as a linear combination of the two components of spinor $\eta_{a\alpha}$. Therefore, the mean-field edge state contains only *two* independent branches of chiral fermions. Obviously the mean-field edge effective theory has $\mathrm{SU}(2)_{\mathrm{color}}$ and $\mathrm{SU}(2)_{\mathrm{spin}}$ symmetries generated by $\tau^l \otimes \hat{\tau}^0$ and $\tau^0 \otimes \hat{\tau}^l$, $l = 1, 2, 3$, respectively. Having the $\mathrm{SU}(2)_{\mathrm{color}} \times \mathrm{SU}(2)_{\mathrm{spin}}$ symmetry and a central charge $c = 2$ (*i.e.*, two branches of chiral fermions), we find that the mean-field edge state is described by $\mathrm{SU}(2)_{\mathrm{color}}$ $\times \mathrm{SU}(2)_{\mathrm{spin}}$ Kac-Moody current algebra of level 1. After including the $\mathrm{SU}(2)_{\mathrm{color}}$ gauge fluctuations to go beyond the mean-field theory, the edge effective theory becomes

$$\lambda^\dagger_{a\alpha} i \left[(\partial_t - iA_0) - v(\partial_x - iA_x)\right]_{ab} \lambda_{b\alpha} \,. \qquad (14)$$

The effect of $\mathrm{SU}(2)_{\mathrm{color}}$ gauge fields is to remove the $\mathrm{SU}(2)_{\mathrm{color}}$ sector of the Kac-Moody algebra from the low energy spectrum.[18] Thus the physical edge state of the p_{x+iy}-state is described by a level-1 $\mathrm{SU}(2)_{\mathrm{spin}}$ Kac-Moody algebra. The physical edge state is identical to the right moving sector of spin-1/2 chain. Despite the finite gap in the bulk, the spin excitation is gapless at the edge. The operator that creates the gapless spin-1/2 quasiparticle on the edge is given by the spin-1/2 primary field $V_\alpha(x,t)$ in the $\mathrm{SU}(2)_{\mathrm{spin}}$ Kac-Moody algebra which has a scaling dimension 1/4. The quasiparticle propagator has a form $\langle V_\alpha(x,t)V_\alpha(0)\rangle \sim (x - vt)^{-1/2}$. The boson operator ϕ_m on the edge becomes the spin-1 primary field which is the spin current operator on the edge. The boson propagator on the edge is given by $\langle \phi_m(x,t)\phi^\dagger_m(0)\rangle \sim (x - vt)^{-2}$. (The boson propagator is short ranged in the bulk due to the finite spin gap.) This will lead to a non-linear I-V curve $I \propto |V|^2 V$ for boson tunneling between two edges. The spin-1/2 quasiparticles can tunnel between two edges separated by a bulk p_{x+iy}-state. The tunneling I-V curve has a form $I \propto |V|^{-1} V$ in the weak tunneling limits. Finally, we briefly mention that the polar condensate has gapless spin excitations in the bulk whereas the p_{x+iy} spin-singlet superfluid has a gapless spin excitation only at the edge. A dramatic difference can be seen in the spin susceptibility by NMR experiments.

e. Conclusions A two-dimensional boson gas in ultra-cold alkali atomic systems can be strongly correlated. A spin-1 boson system can have a very interesting spin-singlet superfluid state, which carries a non-trivial topological order.[21] Such possibility is interesting, since they might exhibit some of the spin liquid phases that have been long theoretically speculated in the context of high T_c superconductors but never been clearly identified by experiments. We believe that the alkali atomic gases may provide the first controlled laboratory to check those speculated theories and enrich our understanding of the strongly correlated systems. In fact the p_{x+iy}-state is closely related to the chiral spin state.[22] The two states have the same bulk effective theory described by level-1 $\mathrm{SU}(2)_{\mathrm{color}}$ Chern-Simons theory,[23] and the same edge effective theory described by level-1 $\mathrm{SU}(2)_{\mathrm{spin}}$ Kac-Moody algebra.[18] The spin sector of the two states are identical.

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