Gapless boundary excitations in the quantum Hall states and in the chiral spin states *

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ABSTRACT: Using the gauge invariances, we show that the (fractional and integral) Quantum Hall (QH) states and the chiral spin states must have gapless boundary excitations. The dynamical properties of those gapless excitations are studied. Under some general assumptions, the gapless excitations are shown to form a representation of the $U(1)$ or $SU(2)$ Kac-Moody algebras and to contribute to a specific heat with a linear temperature dependence. The low energy effective theories for those gapless excitations are derived. The quantum numbers of the gapless boundary excitations are also discussed. In particular, the charge zero sector of the low lying boundary excitations in the fractional QH states are shown to be described by the charge zero sector of free fermions with fractional charges.

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I. INTRODUCTION

Recently it is shown that a rigid state (a state with no gapless quasi-particle excitations) may have many non-trivial properties even at energies below the energy gap.[1,2,3] Although there are no particle-like local excitations at those energies, global excitations may exist. The global excitations appear in the form of ground state degeneracies. The ground state degeneracies may depend on the topologies of the compactified spaces. In this case we say that the rigid state contains non-trivial topological orders. The Fractional Quantum Hall (FQH) states and the recently proposed chiral spin states[4,5] are two examples of topologically ordered states.[1,2,3] Characterization of the topological orders is discussed in Ref. 1 and 2. The topological order can be partially characterized by the ground state degeneracies of the rigid state on compactified spaces. It can also be characterized more completely by non-Abelian Berry's phases generated by a family of twisted Hamiltonians. Arovas and Haldane[6] recently introduce a spin twist operation to spin Hamiltonians. The spin twist is used to probe the topological structures in the ground states of a spin system. They show that the topological structures in the spin liquid states can be (partially) characterized by the Chern number associated with a family of the spin-twisted ground states.

In Ref. 1,2,3 and 6, the topologically ordered states are studied on the compactified space. However, experimentally, it is very difficult to compactify a two dimensional lattice into, say, a torus. The results in Ref. 1,2,3 and 6 probably can only be tested by numerical simulations on computers. In this paper we are going to study the FQH states and the chiral spin states on spaces with boundaries. We will show that the FQH states and the chiral spin states have gapless boundary excitations. In the FQH states the *neutral* gapless boundary excitations are described by the charge zero sector of free (chiral) fermion theories which in general contain several branches of fermions. The fermions in each branch carry electrical charges q_I satisfying

$$\sum_I \frac{c_I}{|c_I|} q_I^2 = (\sigma_{xy} - \sigma'_{xy})h = (\nu - \nu')e^2 \tag{1}$$

where σ_{xy} and σ'_{xy} (ν and ν') are the Hall conductances (the filling fractions) on the two sides of the boundary, I labels different kinds of the (chiral) fermions and c_I is the velocity of the fermions. The vacuum has $\sigma_{xy} = \nu = 0$. In the chiral spin state, the gapless boundary excitations are shown to form a representation of several independent $SU(2)$ Kac-Moody algebras.[7] The levels l_I of those Kac-Moody algebras satisfy

$$\sum_I \frac{c_I}{|c_I|} l_I = \frac{(k - k')}{2} \tag{2}$$

where k and k' are the levels of the chiral spin states on the two sides of the boundary. The vacuum has $k = 0$. The above results are the consequences of the $U(1)$ gauge invariance of the electromagnetic field and the $SU(2)$ gauge invariance of the spin twist introduced by Arovas and Haldane.[6]

Halperin[7] has studied the boundary excitations of the integral quantum Hall (IQH) states and found gapless boundary excitations. He shows that the boundary excitations remain gapless even in presence of weak random potentials. Our results are generalizations of Halperin's results to the FQH states and the chiral spin states. We would like to emphasize that Halperin's results can be obtained by perturbing around a free electron system. However the electrons in the FQH states and the chiral spin states are strongly correlated. In this case it is even not clear whether the boundary excitations are described by the Fermi liquid theory or not, especially after knowing that the boundary excitations may carry fractional charges. New approach is needed to study the boundary excitations of a strongly correlated electron system. In this paper we will use the gauge invariance and the locality of the theory to study some general dynamical properties of the boundary excitations in the FQH states and in the chiral spin states.

The paper is arranged as the following. In Section 2 we study the dynamics of the boundary excitations in the FQH states. The gapless boundary excitations are shown to be described by the $U(1)$ Kac-Moody algebras. The $U(1)$ Kac-Moody algebras are equivalent to the charge zero sector of free (chiral) fermions. (The charge zero sector is formed by states with zero total charge. Those states correspond to particle-hole excitations.) Thus the *neutral* boundary excited states in the FQH states can be described by the charge zero sector of the Fermi liquid theory. However, as required by the gauge invariance, the fermions in such Fermi liquids must carry fractional charges which satisfy the sum role (1). In Section 3 we review the Arovas and Haldane's spin twist and the associated $SU(2)$ gauge symmetry. The boundary excitations of the chiral spin states are shown to be described by the $SU(2)$ Kac-Moody algebras. Those boundary excitations carry non-zero spins. In Section 4 we study the stability of the gapless boundary excitations. We conclude the paper in Section 5.

II. BOUNDARY EXCITATIONS IN THE FQH STATES

Following Ref. 8 and 9 we can give a simple argument that the FQH states must have gapless boundary excitations. Consider a QH state with a filling fraction ν on an annulus with inner radius r_1 and outer radius r_2 (Fig. 1). Adding a unit flux quantum $\Phi_0 = hc/e$ to the hole moves $\nu e = \sigma_{xy}\frac{h}{e}$ amount charge from, say, the inner boundary to the outer boundary. The work required to add a unit flux is zero in the thermodynamic limit ($r_1 \to \infty, r_2 \to \infty$). Therefore the boundary excitation which transfers νe charge from one boundary to another is gapless.

The existence of gapless boundary excitations is a simple and almost trivial consequence of the gauge invariance. However, the more important questions are what are the dynamics and the quantum numbers of the gapless boundary excitations. Experimental tests of the gapless excitations rely on those properties. The main purpose of this paper is to determine the dynamics and the quantum numbers of the gapless excitations. In this section we will concentrate on the FQH states (and the IQH states). In Section 3 we will use the Arovas and Haldane's spin twist to study the chiral spin states.

Assume that a two-dimensional electron system demonstrates the FQH effect (or the IQH effect) in a background magnetic field $\bar{A}_i$ ($\bar{A}_0 = 0$) with a Hall conductance $\sigma_{xy} = \nu\frac{e^2}{h}$. After integrating out the electrons, we obtain an effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{\nu e^2}{4\pi}\delta A_\mu \partial_\nu \delta A_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{4g_1^2}(\delta F_{0i})^2 - \frac{1}{4g_2^2}(\delta F_{12})^2 + \dots \tag{3}$$

where $\delta A_\mu = A_\mu - \bar{A}_\mu$ and $\delta F_{\mu\nu} = \partial_\mu \delta A_\nu + \partial_\nu \delta A_\mu$ is the field strength. The coefficient of the Chern-Simons term $\delta A_\mu \partial_\nu \delta A_\lambda \epsilon^{\mu\nu\lambda}$ is given by the quantized Hall conductance.

On a compactified space, the action $S_{\text{bulk}} = \int d^3x \mathcal{L}_{\text{eff}}(\delta A_\mu)$ is invariant under the gauge transformation. However, on a space with boundary, say, a disc D, S_{bulk} is *not* invariant:

$$S_{\text{bulk}}(\delta A_\mu + \partial_\mu f(x)) - S_{\text{bulk}}(\delta A_\mu) = \int dx_0 d\sigma \frac{\nu e^2}{4\pi} f \ \delta F_{\sigma 0} \tag{4}$$

where σ parametrizes the boundary of the disc. Because the microscopic theory is gauge invariant, (4) implies that S_{bulk} is not the complete action of the FQH states on the disc. Since the change in S_{bulk} is just a boundary term, the total gauge invariant effective action may be obtained by including a boundary action associated with the boundary excitations $S_{\text{tot}} = S_{\text{bulk}} + S_{\text{bd}}$. Under the gauge transformation S_{bd} should transform as

$$S_{\text{bd}}(\delta A_\mu + \partial_\mu f) - S_{\text{bd}}(\delta A_\mu) = -\int dx_0 d\sigma \frac{\nu e^2}{4\pi} f \delta F_{\sigma 0} \tag{5}$$

so that S_{tot} is gauge invariant. The boundary effective action satisfying (5) has a form:

$$S_{\text{bd}} = \int dt d\sigma dt' d\sigma' \frac{1}{2} \delta A_\alpha(t,\sigma) K^{\alpha\beta}(t-t', \sigma-\sigma') \delta A_\beta(t', \sigma') \tag{6}$$

where $\alpha, \beta = 0, \sigma$ and $t = x_0$. $K^{\alpha\beta}$ is the current-current correlation function of the boundary excitations $K^{\alpha\beta}(t,\sigma) = i\langle 0|T(j^\alpha(t,\sigma) j^\beta(0,0))|0\rangle$, which satisfies (in k-space)

$$\begin{aligned} -k_\alpha K^{\alpha\beta} &= \frac{\nu e^2}{4\pi} \epsilon^{\alpha\beta} k_\alpha \\ K^{\alpha\beta}(k_\alpha) &= K^{\beta\alpha}(-k_\alpha) = K^{\alpha\beta *}(-k_\alpha) \end{aligned} \tag{7}$$

where $k_0 = \omega$ is the frequency, $k_\sigma = k = \frac{2\pi n}{L}$ is the momentum in σ direction, and L is the length of the boundary. We see that the complete action of the FQH states on a disc is given by S_{bulk} describing the bulk excitations *plus* S_{bd} describing the boundary excitations. S_{bulk} and S_{bd} separately are not gauge invariant. But the total action S_{tot} is gauge invariant.

The effective action (6) is obtained by integrating out boundary excitations. The current-current correlation function of those boundary excitations should satisfy (7) in order for the total action S_{tot} to be gauge invariant. In the following we would like to use (7) to study the dynamical properties of the low lying boundary excitations. We will show that the boundary excitations are described by chiral Kac-Moody current algebras.

In field theory the relation between three dimensional Chern-Simons theories and two dimensional chiral conformal theories was first pointed out by Witten.[10] The mathematical formalism and the ideas behind our calculations are similar and motivated from those in Ref. 10. However the physical problem studied in this paper has some fundamental differences than that in Ref. 10. A) The gauge fields in our problem are background fields. They do not fluctuate and have no dynamics. While the gauge fields in Ref. 10 are dynamical fluctuating fields. The gapless boundary excitations in our theory come from electrons. Those excitations exist even when the gauge fields are fixed. The gapless boundary excitations in the FQH states may have several branches even when we have only one gauge field. The two dimensional conformal theories studied Ref. 10 come from the dynamics of the gauge fields. Each gauge fields only give rise to one Kac-Moody algebra. B) Although the Hilbert space of the low lying boundary excitations in our problem can be shown to form a representation of the Kac-Moody algebra, the Hamiltonian which governs the dynamics of the boundary excitations does not respect conformal symmetry. The Hamiltonian even does not have Lorentz symmetry since different edge branches in general have different velocities. The theories studied in Ref. 10 all respect conformal symmetries. One must be careful about which results in Ref. 10 are related to QH states and chiral spin states and which are not. Our approach is also closely related to that in Ref. 11 where the relation between the gauge anomaly in $2N$ dimensions and the Chern-Simons term in $2N+1$ dimensions are studied.

In this paper one of the main problems concerning us is the following: without knowing the symmetry of the electron Hamiltonian we would like to obtain, as much as possible, the dynamical properties of the boundary excitations from the gauge invariance and the locality of the theory. It is well known in field theory that the gauge anomaly (4) can be cancelled by chiral fermions on the boundary. What we try to do here is the reverse. We would like to show that in order to cancel the anomaly, the boundary excitations must be described by chiral fermions (or more precisely, chiral $U(1)$ Kac-Moody algebras). Our derivation is not completely general. It depends on some weak assumptions which will be summaried at the end of this section.

The current-current correlation $K^{\alpha\beta}$ can be written as

$$K^{\alpha\beta}(t,\sigma) = i\sum_{n,k} e^{i\omega_{nk}|t|}\Big[e^{-ik\sigma}\langle 0|j^\alpha(t,\sigma)|k;n\rangle\langle k;n|j^\beta(0)|0\rangle\theta(t) + e^{ik\sigma}\langle 0|j^\beta(0)|k;n\rangle\langle k;n|j^\alpha(t,\sigma)|0\rangle\theta(-t)\Big] \tag{8}$$

where ω_{nk} is the energy of the state $|k;n\rangle$. In the k-space

$$K^{\alpha\beta}(\omega,k) = L\sum_n \frac{f^{\alpha *}_{n,k}f^\beta_{n,k}}{\omega-\omega_{n,k}+i\delta} - \frac{f^{\beta *}_{n,-k}f^\alpha_{n,-k}}{\omega+\omega_{n,-k}-i\delta} \tag{9}$$

where $\delta = 0^+$ and

$$f^\alpha_{nk} \equiv \langle k;n|j^\alpha(t=0,\sigma=0)|0\rangle. \tag{10}$$

Assume all the boundary excitations have finite energy gap $\omega_{n,k} \geq \Delta > 0$, then $K^{\alpha,\beta}(\omega,k)$ is a smooth function of ω near $\omega = 0$. If we further assume that the theory is local, $K^{\alpha\beta}(\omega,k)$ should be a smooth function of k near $k = 0$ (*e.g.,* $K^{\alpha\beta}(\omega,k)$ can not behave like $\frac{F^{\alpha\beta}(\omega)}{k}$ near $k=0$). However one can easily check that a smooth function of ω and k (near $\omega = 0$ and $k = 0$) can never satisfy (7). Therefore, for local theories, the condition (7) implies the existence of gapless boundary excitations.

Let us assume the boundary excitations have many branches labeled by I and $K^{\alpha\beta}$ have poles at $\omega = c_I k$ where c_I is the velocity of the Ith branch. In this case (9) can be rewritten as (for small (ω,k))

$$K^{\alpha\beta}(\omega,k) = \sum_I \left[\frac{\gamma^{\alpha\beta}_{I,k}}{\omega-c_Ik+i\delta} - \frac{\gamma^{\beta\alpha}_{I,-k}}{\omega-c_Ik-i\delta}\right] + P(\omega,k) \tag{11}$$

where $P(\omega, k)$ is a polynomial of ω and k which comes from the gapful excitations. Note that $\gamma^{\alpha\beta}_{I,k}$ is non-zero only when $c_I k \geq 0$. This is because the excited states always have positive energies and there are no states with $\omega_{n_I,k} = c_I k < 0$. Plugging (11) into (7) and using the fact that $\gamma^{\alpha\beta}_{I,k}$ only depend on k, we find that $K^{\alpha\beta}$ must have a form (up to a polynomial in ω and k)

$$K^{\alpha\beta} = \begin{cases} -\sum_I \frac{k\eta_I}{\omega - c_I k} \ , & (\alpha,\beta) = (0,0) \\ -\frac{1}{2}\sum_I \frac{\omega + c_I k}{\omega - c_I k}\eta_I \ , & (\alpha,\beta) = (\sigma,0),(0,\sigma) \\ -\sum_I \frac{c_I \omega \eta_I}{\omega - c_I k} \ , & (\alpha,\beta) = (\sigma,\sigma) \end{cases} \tag{12}$$

for small ω and k, where

$$\sum_I \eta_I = \frac{\nu e^2}{2\pi}. \tag{13}$$

From (11), (12) and the fact that $\gamma^{00}_{I,k} \geq 0$, we find that $c_I \eta_I < 0$.

Each term in the summations in (12) arises from gapless boundary excitations with velocity c_I. It is convenient to write the current j^α as a summation of j^α_I:

$$j^\alpha = \sum_I j^\alpha_I \tag{14}$$

such that

$$K^{\alpha\beta}_{IJ} \equiv \langle j^\alpha_I j^\beta_J \rangle = \delta_{IJ} \begin{cases} -\frac{k\eta_I}{\omega - c_I k} \ , & (\alpha,\beta) = (0,0) \\ -\frac{1}{2}\frac{\omega + c_I k}{\omega - c_I k}\eta_I \ , & (\alpha,\beta) = (\sigma,0),(0,\sigma) \\ -\frac{c_I \omega \eta_I}{\omega - c_I k} \ , & (\alpha,\beta) = (\sigma,\sigma) \end{cases} \tag{15}$$

Therefore j^α_I is associated with the gapless excitations with velocity c_I. $j^\alpha_{I,k}$ generates a state with an energy $\omega_k = c_I k$:

$$H j^\alpha_{I,k}|0\rangle = c_I k j^\alpha_{I,k}|0\rangle \tag{16}$$

where $j^\alpha_{I,k} = \int d\sigma \frac{1}{\sqrt{L}} e^{i\sigma k} j^\alpha_I(\sigma)$. From (15) we can obtain the vacuum expectation values of the commutator We find that

$$\begin{aligned} \langle 0| \left[j^+_{I,k'}, j^+_{J,k} \right] |0\rangle &= |\eta_I| k \delta_{k+k'} \delta_{IJ} \\ \langle 0| \left[j^-_{I,k'}, j^+_{J,k} \right] |0\rangle &= \langle 0| \left[j^-_{I,k'}, j^-_{J,k} \right] |0\rangle = 0. \end{aligned} \tag{17}$$

where $j^\pm_I = \frac{1}{2}(j^0 \pm \frac{1}{c_I} j^\sigma)$.

Under certain assumptions (*e.g.,* the locality of the theory) we can further show that (see appendix) (16) and (17) imply the operator equations

$$\left[H, j^\alpha_{I,k} \right] = c_I k j^\alpha_{I,k} \tag{18}$$

and

$$\begin{aligned} \left[j^+_{I,k'}, j^+_{J,k}\right] &= |\eta_I| k \delta_{k+k'} \delta_{IJ} \\ \left[j^+_{I,k'}, j^-_{J,k}\right] &= \left[j^-_{I,k'}, j^-_{J,k}\right] = 0 \end{aligned} \tag{19}$$

in the subspace of states with small momentum k ($k\xi \ll 1$, where ξ is the magnetic length) and in the limit $L \to \infty$. Therefore the low lying boundary excitations form a representation of several independent chiral $U(1)$ Kac-Moody algebras.[12] The current algebra (19) determines the Hilbert space and (18) determines the dynamics of the low lying boundary excitations.

The properties of Kac-Moody algebra (19) is well known.[12] For completeness we will give a brief review. A single copy of chiral $U(1)$ Kac-Moody algebra is given by the following current algebra:

$$\begin{aligned} \left[j^+_{k'}, j^+_k\right] &= |\eta| k \delta_{k+k'} \\ \left[j^+_{k'}, j^-_k\right] &= \left[j^-_{k'}, j^-_k\right] = 0 \\ \left[H, j^\pm_k\right] &= ck j^\pm_k \end{aligned} \tag{20}$$

(20) corresponds to a half of the Tomonaga model with fermions moving only in one direction.[12] An irreducible representation of the above algebra can be obtained from the vacuum state $|0\rangle$ satisfying

$$\begin{aligned} j^+_k |0\rangle &= 0\Big|_{ck\leq 0} \\ j^-_k |0\rangle &= 0\Big|_{\forall k} \end{aligned} \tag{21}$$

(Note if $j^+_k|0\rangle \neq 0$ and $ck < 0$, the state $j^+_k|0\rangle$ would have negative energy.) The Hilbert space (denoted as $\mathcal{H}_0$) of the irreducible representation is generated from the vacuum state $|0\rangle$ by the operators j^+_k with $ck > 0$. Within the Hilbert space $\mathcal{H}_0$, the operator $j^-_k = 0$ (this can also be obtained from the current conservation law $\partial_0 j^0 + \partial_\sigma j^\sigma = 0$). From (20) and (21) we see that the pair (j^+_k, j^+_{-k}) generates a harmonic oscillator with frequency $\omega_k = |ck|$. Because (j^+_k, j^+_{-k}) and $(j^+_{k'}, j^+_{-k'})$ commute if $k \neq k'$, the Hilbert space $\mathcal{H}_0$ of the irreducible representation is a direct product of the Fock spaces of the harmonic oscillators generated by (j^+_k, j^+_{-k}). The specific heat coming from these low lying excitations is $C = \frac{\pi}{6}\frac{T}{c}\, L$.

We would like to emphasize that the arguments in Ref. 9,8 only show the existence of gapless excitations. This by no means implies that the specific heat has a linear temperature (T) dependence, since the specific heat depends on the number of states at low energies. However, if the theory is local, one can show that the number of states at low energies is such that the specific heat from the gapless excitations with finite velocity is linear in T.

For a generic situation, the low lying boundary excitations may form a representation of several independent $U(1)$ Kac-Moody algebras (19). The specific heat in this case is given by $C = \sum_I \frac{\pi}{6}\frac{T}{c_I}\, L$. The above is the bosonic construction of the boundary states. Due to the boson-fermion equivalence in 1+1 dimensions, the representation of (19) can be constructed from a fermion theory[13]

$$S_{\text{bd}} = \int dx_0 d\sigma \sum_I i\psi_I^\dagger \left[(\partial_0 + iq_I \delta A_0) + c_I(\partial_\sigma + iq_I\delta A_\sigma)\right]\psi_I \tag{22}$$

where ψ_I are the chiral fermion fields. One can show explicitly that the electric current in (22), $(j_I^0, j_I^\sigma) = (q_I\psi_I^\dagger\psi_I, c_I q_I \psi_I^\dagger \psi_I)$, obey the Kac-Moody algebra (19) with the "central charge"

$$|\eta_I| = \frac{q_I^2}{2\pi}. \tag{23}$$

One can also show that the current-current correlation of (22) is given by (12) through a direct calculation of the diagram in Fig. 2. Thus (13) implies that the electric charge q_I of the boundary excitations satisfy

$$\sum_a \frac{c_I}{|c_I|} q_I^2 = \nu e^2 \tag{24}$$

It has been shown that the charge zero states of (22) form the irreducible representation of the Kac-Moody algebra (19).[12]

The above results can be easily generalized to the boundaries which separate two quantum Hall states with the Hall conductances σ_{xy} and σ'_{xy}. In this case (24) is replaced by (1).

We would like to remark that in the above we only show that the charge zero sector of the boundary excitations is described by the charge zero sector of (22). We did not say any thing about the charged excited states. The charged excited states on the boundary of the FQH states may not be described by the charged excited states in (22). Especially the total charge of a boundary excited state may not be a multiple of q_I. In fact one can write down one dimensional models[12] which form a representation of the $U(1)$ Kac-Moody algebra (20) with central charge $|\eta| = q^2/2\pi$. But the total charges of the charged excited states in the models are not multiples of q. Thus (22) should be used with caution.

In this paper when we speak of charge of the boundary excitations, we realy mean the charge measured by the current correlation function (see (23)). Such a charge will be called optical charge. The total charges of boundary excited states are not necessarily multiples of the optical charge (measured by the current correlation function). The sum rule (1) is realy satisfied by the optical charges of the boundary excitations.

We would like to emphasize that the results in this section (and in Section 3) rely on two assumptions:

A) The states *generated by the current* j^α have *discrete* velocities in the low energy limit. The energies and the momenta (ω, k) of the states generated by j^α lie within the shaded region in Fig. 3a. Our results do not apply to the situations where the states generated by j^α have a continuous distribution of velocities (Fig. 3b). Our assumption would be reasonable if we could show that for a generic interacting theory the velocities of the states generated by j^α would in general discretize in the infrared limit. Although we are unable to show correctness of the above conjecture here, we do have an example which supports the conjecture. The example is an one dimensional boson system. For free bosons, the low lying excitations (generated by j^α) have a continuous distribution of velocities. However, after we turn on the interactions between bosons, the low lying excitations of the system are phonons with discrete velocities $\pm c$.

B) The total current j^α can be written as a sum (14) and the current operator j_I^α generating the states with velocity c_I is a *local* operator. Notice that the operator j_I^α generates a pair of the quasi-particle and the quasi-hole. Although the operator creates a *single* quasi-particle (or quasi-hole) must be non-local in terms of the original electron operators (since the quasi-particles carry fractional charges), the current operator may still be a local operator.

III. BOUNDARY EXCITATIONS OF THE CHIRAL SPIN STATES

Now let us consider the boundary excitations of the chiral spin states. First we would like to review Arovas and Haldane's spin twist[6] in terms of the mean field theory of the chiral spin states. [4]

The spin twist is introduced by generalizing the Heisenberg model to

$$H = \sum J_{ij} \vec{S}_i \cdot U_{ij} \cdot \vec{S}_j + \sum A_0^a(i) S_i^a \qquad a = 1, 2, 3 \tag{25}$$

where U_{ij} is a 3×3 matrix

$$U_{ij} = e^{iA_{ij}^a T^a} \quad , \quad A_{ij}^a = -A_{ji}^a, \qquad a = 1, 2, 3 \tag{26}$$

and T^a are the generators of $SO(3)$ Lie algebra satisfying

$$[T^a, T^b] = i\epsilon^{abc} T^c \tag{27}$$

(25) can be put into a Lagrangian form with the help of electron operator $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$:

$$\begin{aligned} L = &- i \sum c_i^\dagger \left(\partial_t + i\frac{1}{2} A_0^a \sigma^a \right) c_i - \sum J_{ij} \left(c_i^\dagger e^{i\frac{1}{2} A_{ij}^a \sigma^a} c_j \right) \left(c_j^\dagger e^{i\frac{1}{2} A_{ji}^a \sigma^a} c_i \right) \\ &- \sum \lambda_i (c_i^\dagger c_i - 1) \end{aligned} \tag{28}$$

where λ_i is the Lagrange multiplier to enforce the constraint $c_i^\dagger c_i = 1$ and the spin $\vec{S}$ is related to the electron operator c by $\vec{S} = \frac{1}{2} c^\dagger \vec{\sigma} c$. The partition function of (28) is a functional of $A_0^a(i)$ and A_{ij}^a:

$$\begin{aligned} Z(A_0^a, A_{ij}^a) &= \int Dc^\dagger Dc D\lambda e^{i\int L(c, A^a, \lambda) dt} \\ &\equiv e^{iS_0(A_0^a, A_{ij}^a)}. \end{aligned} \tag{29}$$

Because L in (28) is invariant under the $SU(2)$ gauge transformation

$$c_i \to c_i' = U_i c_i \tag{30}$$

$$\frac{1}{2}A_0^a(i)\sigma^a \to \frac{1}{2}A_0^a{}'(i)\sigma^a = \frac{1}{2}U_i A_0^a(i)\sigma^a U_i^\dagger - iU_i\partial_t U_i^\dagger \tag{31}$$

$$e^{i\frac{1}{2}A_{ij}^a\sigma^a} \to e^{i\frac{1}{2}A_{ij}^a{}'\sigma^a} = U_i e^{i\frac{1}{2}A_{ij}^a\sigma^a} U_j^\dagger \tag{32}$$

the effective action $S_0(A^a)$ in (29) should also respect the gauge symmetry (31), (32).

The mean field Lagrangian of the chiral spin states is obtained from (28) by replacing[4] $c_i^\dagger e^{i\frac{1}{2}A_{ij}^a\sigma^a} c_j$ by $\chi_{ji}e^{ia_{ji}}$ and λ_i by $\bar\lambda + a_0$:

$$L_{\text{mean}} = \sum -ic_i^\dagger \left(\partial_t + ia_0 + i\frac{1}{2}A_0^a\sigma^a\right) c_i - \sum \bar\lambda c_i^\dagger c_i - \sum J_{ij}\chi_{ij} c_i^\dagger e^{i\frac{1}{2}A_{ij}^a\sigma^a + ia_{ij}} c_j \tag{33}$$

a_{ij} and a_0 are fluctuations around the mean field solution $\bar\lambda$ and χ_{ij}. After integrating out electron field c, we obtain the effective Lagrangian of the chiral spin states $\mathcal{L}_{\text{eff}}(a, A^a)$. The effective Lagrangian $\mathcal{L}_{\text{eff}}(a, A^a = 0)$ is calculated in Ref. 4. Notice that when only $A^3 \neq 0$ the spin up and spin down electrons in (33) decouple. Generalizing the calculation in Ref. 4 and using the $SU(2)$ gauge symmetry we obtain

$$\mathcal{L}_{\text{eff}}(a_\mu, A_\mu) = \frac{k}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} + \frac{k}{8\pi}\text{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right)\epsilon^{\mu\nu\lambda} \tag{34}$$

where A_μ is a 2×2 matrix: $A_\mu = \frac{1}{2}A_\mu^a\sigma^a$ and k is an integer called the level of the chiral spin states. (34) indicates that the level of a chiral spin state can be measured by using the spin twist.

Strictly speaking the effective action

$$S_{\text{eff}}[a_\mu, A_\mu] = \int d^3x \mathcal{L}_{\text{eff}}(a_\mu, A_\mu) \tag{35}$$

is invariant under the gauge transformation

$$A_\mu \to U A_\mu U^\dagger - iU\partial_\mu U^\dagger \equiv A_\mu^U \tag{36}$$

only when the space is compact and k is an *even* integer.[14] From this we obtain an important result that the spin singlet chiral spin states always have *even* levels.

On a space with boundary, say, a disc D, the action $S_0[A_\mu]$ is not gauge invariant:

$$S_0[A_\mu^U] - S_0[A_\mu] \simeq \int dt d\sigma \frac{k}{4\pi}\text{Tr}\ \Lambda\partial_\alpha A_\beta \epsilon^{\alpha\beta}$$
$$U = e^{i\Lambda}. \tag{37}$$

We have assumed that Λ and A_μ are small and only kept the term linear in Λ and A_μ.

Let us introduce the $SU(2)$ boundary current $J^{a\alpha}$ which couples to the $SU(2)$ gauge field through $A_\alpha^a J^{a\alpha}$. Using a similar approach used in section 2, we may write the $SU(2)$ current of the boundary excitations as

$$J^{a\alpha} = \sum_I j_I^{a\alpha}, \tag{38}$$

where $j_I^{\alpha a}$ generates boundary states with velocity c_I and $a = 1, 2, 3$ is the $SU(2)$ vector indices. Notice that within linear response theory, A_α^a in (37) correspond to three independent $U(1)$ gauge field. In this case we may repeat the analysis in section 2 to determine the current correlation function on the edge. We find that in order for the boundary excitations to restore the $SU(2)$ gauge invariance of the total action, the boundary currents $j_I^{\alpha a}$ must satisfy

$$\begin{aligned} \langle 0| \left[j_{I,k'}^{a+}, j_{J,k}^{b+} \right] |0\rangle &= \frac{l_I}{4\pi} \delta_{IJ} \delta^{ab} k \delta_{k+k'} \\ \langle 0| \left[j_{I,k'}^{a+}, j_{J,k}^{b-} \right] |0\rangle &= \langle 0| \left[j_{I,k'}^{a-}, j_{J,k}^{b-} \right] |0\rangle = 0 \end{aligned} \tag{39}$$

where

$$j_{I,k}^{a\pm} = j_{I,k}^{a0} \pm \frac{1}{c_I} j_{I,k}^{a\sigma}$$

and $j_{I,k}^{a\alpha} = \int d\sigma \frac{1}{\sqrt{L}} e^{i\sigma k} j_I^{a\alpha}(\sigma)$. l_I in (39) must satisfy

$$\sum_I l_I \frac{c_I}{|c_I|} = \frac{1}{2} k \tag{40}$$

such that the boundary excitations may cancel the gauge non-invariance (37). k in (40) is the coefficient of the Chern-Simons term in (¿¿3.11¡¡) (*i.e.,* the level of the chiral spin state). Assuming the currents $j_I^{\pm a}(\sigma)$ are local operators, we can show (see appendix) that (40) implies an operator relation

$$\begin{aligned} \left[j_{I,k'}^{a+}, j_{J,k}^{b+} \right] &= \left(\frac{1}{\sqrt{L}} \epsilon^{abc} j_{I,k+k'}^{c+} + \frac{l_I}{4\pi} k \delta_{k+k'} \delta^{ab} \right) \delta_{IJ} \\ \text{others} &= 0 \end{aligned} \tag{41}$$

where $k = \text{integer} \times 2\pi/L$. Because $j_{I,k}^{a\pm}$ generates states with a fixed velocity c_I we can also show that

$$\left[H, j_{I,k}^{a\pm} \right] = c_I k j_{I,k}^{a\pm} \tag{42}$$

(41) and (42) imply that the boundary $SU(2)$ currents $j_I^{\pm a}$ generate several independent $SU(2)$ Kac-Moody algebras. The gapless boundary excitations form a representation of those algebras. Notice that $\sqrt{L} j_{I,k=0}^{a+}$ defined in (41) is the total spin operators which satisfy the $SU(2)$ Lie algebra. l_I is the level os the $SU(2)$ Kac-Moody algebra which must be an integer. Otherwise (41) has no unitary representations.[7]

It is well known in field theory that the Kac-Moody algebra (41) and (42) plus the sum rule (40) are the sufficient condition for the boundary excitations to cancel the gauge anomaly of the Chern-Simens term (34).[11] What we have shown here is that (40), (41) and (42) are the necessary conditions to restore the gauge symmetry. We show that in order to cancel the gauge anomaly of the Chern-Simons term, the boundary excitations must form a representation of $SU(2)$ Kac-Moody algebras. Our derivation does not rely on any symmetries, in particular the conformal symmetry, of the Hamiltonian. However our approach is not completely general. It relies on the two assumptions which will be summarized at the end of the section 2.

Unlike the $U(1)$ Kac-Moody algebra, the $SU(2)$ algebra (41) and (42) describes an interacting theory. However such a theory is still exactly soluble.[7] The specific heat coming from the excitations in a representation of the $SU(2)$ Kac-Moody algebras (41) and (42) is given by[15]

$$C = L \sum_I \frac{\pi l_I}{2(l_I + 2)} \frac{T}{|c_I|}. \tag{43}$$

The levels l_I of the Kac-Moody algebras should satisfy (40). For the boundary which separates the different chiral spin states with levels k and k', the levels of the Kac-Moody algebras of the boundary excitations should satisfy (2).

Before ending this section we would like to discuss the level $k = 2$ chiral spin state studied in Ref. 4 in more detail. From Ref. 4 we know that the mean field chiral spin state is equivalent to the IQH state with the first Landau level filled by the spin up and the spin down electrons, if we turn off the lattice. Using the results in Ref. 8 we find that the boundary excitations of the *mean field* chiral spin state are described by the following free fermion theory (in the long wave length limit)

$$\mathcal{L} = \sum_{\lambda=1,2} \psi_\lambda^\dagger i(\partial_0 - c\partial_\sigma)\psi_\lambda \tag{44}$$

where the electron field $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ carries the spin $\frac{1}{2}$ and the electric charge e. The excitations described by (44) not only form a representation of the level 1 $SU(2)$ Kac-Moody algebra

$$\left[j_k^{+a}, j_{k'}^{+b} \right] = \frac{1}{\sqrt{L}} \epsilon^{abc} j_{k+k'}^{+c} + \frac{1}{4\pi} k \delta_{k+k'} \delta^{ab} \tag{45}$$

where $j^{+a} = \psi_\lambda^+ \left(\frac{\sigma^a}{2}\right)_{\lambda\lambda'} \psi_{\lambda'}$, they also form a representation of the $U(1)$ Kac-Moody algebra (20) with $j^+ = \psi_\lambda^+ \psi_\lambda$. The boundary excitations described by (44) have strong boundary charge fluctuations. The charge fluctuations are due to the gapless excitations associated with the $U(1)$ Kac-Moody algebra. We know that the charge fluctuations are forbidden in the chiral spin state. The actual spin wave function of the chiral spin state is obtained from the mean field wave function by doing the Gutzwiller projection. The Gutzwiller projection removes all the (unphysical) charge fluctuations in the mean field chiral spin state. Therefore in order to use (44) to describe the actual boundary excitations of the chiral spin state we need to remove all the excitations associated with the charge fluctuations or, equivalently, associated with the $U(1)$ Kac-Moody algebra. Notice that the Hilbert space of (44) can be written as a direct product of a representation of the $l = 1$ $SU(2)$ Kac-Moody algebra and a representation of the $U(1)$ Kac-Moody algebra.[16] The excitations of the $SU(2)$ Kac-Moody algebra and the excitations of the $U(1)$ Kac-Moody algebra are independent. In this case the Gutzwiller projection can be performed by simply dropping those excitations associated with the $U(1)$ Kac-Moody algebra. After performing the Gutzwiller projection to the theory (44), we obtain a theory containing only the $SU(2)$ Kac-Moody algebra. Therefore the boundary excitations of the chiral spin state are described by the level 1 $SU(2)$ Kac-Moody algebra. They can not be described by Fermi liquids of any spin.

We would like to mention that the low lying excitations in the one dimensional Heisenberg model are described by level one $SU(2) \times SU(2)$ Kac-Moody algebra, one $SU(2)$ for right moving excitations the other for left moving excitations. Thus the boundary excitations of the level 2 chiral spin state are equivalent to the left (or right) moving excitations in the Heisenberg spin chain.

According to the mean field theory of the chiral spin states,[4] the total effective action of the level 2 chiral spin state (on a space with boundary) is given by

$$\begin{aligned} S_{\text{eff}} = &\int dx^3 \left[\frac{k}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} + \frac{k}{8\pi} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \epsilon^{\mu\nu\lambda} \right] \\ &+ \int dx_0 d\sigma \sum_{\lambda,\lambda'=1,2} \psi_\lambda^\dagger \left[(\partial_0 \delta_{\lambda\lambda'} + i a_0 \delta_{\lambda\lambda'} + i A_{0,\lambda\lambda'}) - c(\partial_\sigma \delta_{\lambda\lambda'} + i a_\sigma \delta_{\lambda\lambda'} + i A_{\sigma,\lambda\lambda'}) \right] \psi_{\lambda'}. \end{aligned} \tag{46}$$

Note that the A_μ field in (46) is a fixed non-dynamical field representing the spin-spin coupling constants in the twisted spin model. The Gutzwiller projection is realized in the effective theory by integrating out the a_μ gauge field in (46). One can show that the gapless excitations associated with the $U(1)$ Kac-Moody algebra are eaten by the gauge field a_μ and obtain a finite energy gap.[17] The surviving low lying excitations only contain the $SU(2)$ algebra.

From the above discussions, we see that the (level 2) chiral spin state is closely related to a IQH state with the first Landau level filled by spin up and spin down electrons. However the two states have very different properties due to the Gutzwiller projection in the chiral spin state. The boundary excitations of the chiral spin state are described by the $l = 1$ $SU(2)$ Kac-Moody algebra with specific heat $C = \pi LT/6c$. While the boundary excitations in the QH state are described by two chiral fermions which form a $U(1) \times SU(2)$ Kac-Moody algebra. The specific heat is given by $C = 2\pi LT/6c$. The modes associated with the charge fluctuations (the $U(1)$ algebra) is absent in the chiral spin state.

If the spin liquid state in the Cu-O planes of the high T_c superconductors is described by, for example, the level 2 chiral spin state, our results imply that there may be a lot of gapless spin excitations in the Cu-O planes even in the superconducting phase. This is because the chiral spin state in the Cu-O planes in general has a domain structure. The chiral spin state in some domains has $k = +2$ and in others has $k = -2$. There are two gapless excitations on the boundaries (domain walls) which separates two domains with $k = +2$ and $k = -2$. The two gapless excitations are described by two copies of the $SU(2)$ Kac-Moody algebra. These gapless excitations on the domain boundaries contribute to a specific heat with a linear T dependence even in the superconducting state. The velocity of the boundary excitations is expected to be of order of the spin wave velocity, which is much less then typical Fermi velocities in metals. We would like to emphasize that the two gapless excitations on the domain boundaries move in the same direction. In the next section we will argue that the gaplessness of the boundary excitations is protected by the topological order. Weak impurities and interplane interactions can not open a gap for those gapless boundary excitations.

IV. STABILITY OF THE GAPLESS BOUNDARY EXCITATIONS

The stability of the gapless edge excitations is guaranteed by the chiral property of the edge excitations. When all the excitations move in the same direction there cannot be any back scattering no matter what interactions the edge excitations may have. The lack of the back scattering prevent the formation of the energy gap. The gaplessness of the edge excitations is robust against weak perturbations.

To demonstrate the above point we would like to consider the following simple model

$$\begin{aligned}\mathcal{L} = &\psi_I^\dagger \left[i\left(\partial_0 + iq_I A_0\right) + c_I i(\partial_\sigma + iq_I A_\sigma)\right]\psi_I \\ &+ \psi_J^\dagger \left[i(\partial_0 + iq_J A_0) + c_J i(\partial_\sigma + iq_J A_\sigma)\right]\psi_J \\ &+ m(\psi_I^\dagger \psi_J + \psi_J^\dagger \psi_I).\end{aligned} \tag{47}$$

m can be regarded as the effective "mass" term generated by the interactions. The mass term is a relevant perturbation and has a potential to open an energy gap. In order for the mass term to respect the gauge symmetry, ψ_I and ψ_J must carry the same charge, $q_I = q_J$. Since the gauge field in (47) is fixed we may set $A_0 = A_\sigma = 0$. The energy spectrum is given by

$$\epsilon_k^\pm = \frac{c_I + c_J}{2} k \pm \sqrt{\left(\frac{c_I - c_J}{2}\right)^2 k^2 + m^2}. \tag{48}$$

The spectrum $\epsilon_k^\pm$ is plotted in Fig. 4 and it has very different behavior for two cases, $c_I c_J > 0$ and $c_I c_J < 0$. When c_I and c_J have the same sign (*i.e.,* ψ_I and ψ_J are both right movers or left movers), the mass term does *not* open an energy gap (Fig. 4a). Only the Fermi momenta and the velocities of the low lying excitations are modified. When c_I and c_J have the opposite signs, the mass term does open an energy gap $\Delta = 2m$ (Fig. 4b).

V. CONCLUSIONS

In this paper we have shown that the gapless boundary excitations of the QH states and the chiral spin states form a representation of several (chiral) Kac-Moody algebras. The Kac-Moody algebras with different velocities are shown to commute with each other. Using the representation theory of the Kac-Moody algebras, we calculate the partition functions of the gapless boundary excitations. The specific heats are found to have a linear T dependence. The quantum numbers of the gapless boundary excitations are shown to satisfy certain sum rules. We also argue that chiral gapless boundary excitations are stable against small perturbations.

We would like to point out that the results in this paper do not depend on the details of the Hamiltonian. We only require that the planar state has a finite energy gap and the effective Lagrangian contains a Chern-Simons term. Many properties of the boundary excitations can be determined from the gauge invariance and the locality of the theory. Certainly some properties, like the velocities of the boundary excitations, remain undetermined. They have to be calculated from the microscopic Hamiltonian. It seems to me that the number of the branches of the boundary excitations are closely related to the topological orders in the FQH states and the chiral spin states. One may be able to determine the number of the branches of the boundary excitations directly from ground degeneracies and the non-Abelian Berry's phases studied in Ref. 1,2,3.

The existence of the gapless boundary excitations is a characteristic property of the FQH states and the chiral spin states. This property can be tested in experiments.[18] Measuring the quantum numbers carried by the gapless boundary excitations could be a powerful and practical way to probe the topological orders in the parent planar state.

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Note added: After submitting this paper, I obtained a preprint by A. MacDonald[19] in which he also pointed out the existence of the gapless edge excitations in the FQH states. I would like to thank S. Girvin for informing me the MacDonald's paper.

Since Oct. 1989 many new results about the edge excitations in the FQH effects are obtained based on the Kac-Moody algebra approach. In particular we have a very good qualitative understanding of the edge excitations in the hierarchy FQH states and the Jain's FQH states. Those new developments can be found in Ref. 20.

APPENDIX

In this appendix we would like to show that the vacuum expectation values of the commutators (17) imply the operator relations (19) in the subspace of states with small momenta. Our derivation relies on the locality of the theory.

First we would like to show that (18) is valid in the small-momentum subspace. We know that $j^\alpha_{I,k}|0\rangle$ is an eigenstate of H:

$$H\ j^\alpha_{I,k}|0\rangle = c_I k\ j^\alpha_{I,k}|0\rangle \tag{49}$$

provided that the momentum k is small. Let us consider a large wave packet of size l and centered at σ

$$|\psi^{I\alpha}_{\sigma,k}\rangle = \sum_{k'} e^{-(k'-k)^2 l^2 + i\sigma\cdot k'} j^\alpha_{I,k'}|0\rangle. \tag{50}$$

In the limit $lk \gg 1$ we have

$$H|\psi^{I\alpha}_{\sigma,k}\rangle = c_I k|\psi^{I\alpha}_{\sigma,k}\rangle + O\left(\frac{c_I}{l}\right).$$

Now consider a wave function with two wave packets

$$|\psi^{I\alpha}_{\sigma_1,k_1};\psi^{J\beta}_{\sigma_2,k_2}\rangle = \sum_{k'} e^{-(k'-k_1)^2 l^2 + i\sigma_1\cdot k'} j^\alpha_{I,k'} \sum_{k''} e^{-(k''-k_2)^2 l^2 + i\sigma_2\cdot k''} j^\beta_{J,k''}|0\rangle. \tag{51}$$

If the two wave packets do not overlap (*i.e.,* $|\sigma_1 - \sigma_2| \gg l$) and if the interactions in the theory are short ranged, we have

$$H|\psi^{I\alpha}_{\sigma_1,k_1};\psi^{J\beta}_{\sigma_2,k_2}\rangle = (c_I k_1 + c_J k_2)|\psi^{I\alpha}_{\sigma_1,k_1};\psi^{J\beta}_{\sigma_2,k_2}\rangle + O\left(\frac{c_I}{l} + \frac{c_J}{l}\right). \tag{52}$$

Because $|\psi^{I\alpha}_{\sigma_1,k_1};\psi^{J\beta}_{\sigma_2,k_2}\rangle$ are superpositions of states $j^\alpha_{I,k} j^\beta_{J,k'}|0\rangle$ with (k,k') near (k_1,k_2) therefore $j^\alpha_{I,k_1} j^\beta_{J,k_2}|0\rangle$ should also have an energy near $c_I k_1 + c_J k_2$. More precisely

$$H\ j^\alpha_{I,k_1} j^\beta_{J,k_2}|0\rangle = (c_I k_1 + c_J k_2) j^\alpha_{I,k_1} j^\beta_{J,k_2}|0\rangle + O\left(\frac{c_I + c_J}{l}\right). \tag{53}$$

The last term in (53) can be dropped if we take the limit $l \to \infty$. Similarly we can show that

$$H\ j^{\alpha_1}_{I_1,k_1} \dots j^{\alpha_I}_{I_N,k_N}|0\rangle = (c_{I_1}k_1 + \dots + c_{I_N}k_N) j^{\alpha_1}_{I_1,k_1} \dots j^{\alpha_N}_{I_N,k_N}|0\rangle \tag{54}$$

if all momentum $k_1, \dots, k_N$ are small. (54) implies (18) in the small-momentum subspace.

Now we would like to use (18) to show that the currents $j^\alpha_{I,k}$ with different I are independent, *i.e.,*

$$\left[j^\alpha_{I,k}, j^\beta_{J,k'}\right] = 0\ , \qquad I \neq J \tag{55}$$

in the small-momentum subspace. Let us introduce

$$\hat{O}_I(t=0) = \int h_I(\sigma) j^\alpha_I(\sigma) d\sigma \tag{56}$$

where $h_I(\sigma)$ is a smooth function and is non-zero only when $|\sigma| < l$ (here l is a large length scale). Notice that

$$\left[\hat{O}_I(0), \hat{O}_J(0)\right] = e^{iHt}\left[\hat{O}_I(t), \hat{O}_J(t)\right] e^{-iHt} \tag{57}$$

where

$$\hat{O}_I(t) = e^{-iHt}\hat{O}_I(0)e^{iHt} = \int h_I(\sigma - c_I t) j_I^\alpha(\sigma) d\sigma. \tag{58}$$

The last equality in (58) is due to the fact that $j_I^\alpha(\sigma)$ creates states with a fixed velocity c_I. When t is large enough, $h_I(\sigma - c_I t)$ and $h_J(\sigma - c_J t)$ do not overlap (notice $c_I \neq c_J$). Assuming $j_I^\alpha(\sigma)$ are local operators, *i.e.*, $[j_I^\alpha(\sigma), j_J^\beta(\sigma')] = 0$ when σ and σ' are separated, we have

$$\left[\hat{O}_I(t), \hat{O}_J(t)\right] = 0. \tag{59}$$

From (59) and (57) we conclude that (55) is valid in the small-momentum subspace.

Because the current j_k^α generates $U(1)$ symmetry, we have

$$\left[j_{k=0}^\alpha, j_k^\beta\right] = 0. \tag{60}$$

(60) can be rewritten as

$$\left[j_{k=0}^\alpha, j_{I,k}^\beta\right] = \hat{O}_I^{\alpha\beta}(k) \tag{61}$$

$$\sum_I \hat{O}_I^{\alpha\beta}(k) = 0 \tag{62}$$

From (18) we see that

$$\left[H, \hat{O}_I^{\alpha\beta}(k)\right] = c_I k \hat{O}_I^{\alpha\beta}(k). \tag{63}$$

(63) and (62) imply that

$$\hat{O}_I^{\alpha\beta}(k) = 0. \tag{64}$$

From (64), (61) and (55), we find that

$$\left[j_{I,k=0}^\alpha, j_{I,k}^\beta\right] = 0. \tag{65}$$

In the following we would like to use (17), (65), (55) and the locality of the theory to derive the operator relation (19). From (17) we find that the commutator $[j_{I,k}^+, j_{I,k'}^+]$ must have a form

$$\begin{aligned}\left[j_I^+(\sigma), j_I^+(\sigma')\right] = &- |\eta_I| i\delta'(\sigma - \sigma') + \delta'(\sigma - \sigma')\hat{O}_1\left(\frac{\sigma + \sigma'}{2}\right) \\ &+ \delta'''(\sigma - \sigma')\hat{O}_3\left(\frac{\sigma + \sigma'}{2}\right) + \ldots\end{aligned} \tag{66}$$

where the operators $\hat{O}_i$ satisfy

$$\langle 0|\hat{O}_i|0\rangle = 0. \tag{67}$$

From (65) we find

$$\int d\sigma \left[j_I^+(\sigma), j_I^+(\sigma')\right] = 0 \tag{68}$$

which implies

$$\frac{1}{2}\hat{O}_1'(\sigma) + \frac{1}{8}\hat{O}_3'''(\sigma) + \ldots = 0. \tag{69}$$

The operator $\hat{O}_1(\sigma)$ can be solved from (69):

$$\hat{O}_1(\sigma) = \hat{O}_0(\sigma) - \frac{1}{4}\hat{O}_3''(\sigma) - \frac{1}{16}\hat{O}_5''''(\sigma) - \ldots \tag{70}$$

where $\hat{O}_0$ is a constant operator (*i.e.*, independent of σ):

$$\hat{O}_0(\sigma) = \hat{O}_0(\sigma'). \tag{71}$$

Because $j_I^+(\sigma)$ is assumed to be a local operator, the operators $\hat{O}_i$ $(i = 0, 1, \ldots)$ derived from $j_I^+(\sigma)$ are also local operators. A local constant operator with zero vacuum expectation value must be zero. This is because for an arbitrary local operator $\Phi(\sigma_0)$

$$[\hat{O}_0(\sigma), \Phi(\sigma_0)] = [\hat{O}_0(\sigma'), \Phi(\sigma_0)] = 0 \tag{72}$$

if we choose σ' to be far away from σ_0. Therefore, in the Hilbert space generated by local operators, the operator $\hat{O}_0$ must be proportional to the identity operator. Since $\hat{O}_0$ has a zero vacuum expectation value, $\hat{O}_0$ must be zero.

Substituting (70) into (66) we find that

$$\begin{aligned}\left[j_I^+(\sigma), j_I^+(\sigma')\right] = & -|\eta_I| i\delta'(\sigma-\sigma') + \left[\delta'''(\sigma-\sigma')\hat{O}_3\left(\frac{\sigma+\sigma'}{2}\right) - \frac{1}{4}\delta'(\sigma-\sigma')\hat{O}_3''\left(\frac{\sigma+\sigma'}{2}\right)\right] \\ & + \left[\delta'''''(\sigma-\sigma')\hat{O}_5\left(\frac{\sigma+\sigma'}{2}\right) - \frac{1}{16}\delta'(\sigma-\sigma')\hat{O}_5''''\left(\frac{\sigma+\sigma'}{2}\right)\right] \\ & + \ldots \end{aligned} \tag{73}$$

In the momentum space (73) can be written as

$$\begin{aligned} & \langle -k_4|[j_{I,k_1}^+, j_{I,k_2}^+]|k_3\rangle \\ = & \frac{1}{L}\int d\sigma d\sigma' e^{i(k_1\sigma + k_2\sigma')}\Big[-|\eta_I| i\delta'(\sigma-\sigma')\delta_{k_3+k_4} + \delta'''(\sigma-\sigma')\langle -k_4|\hat{O}_3\left(\frac{\sigma+\sigma'}{2}\right)|k_3\rangle \\ & \qquad -\frac{1}{4}\delta'(\sigma-\sigma')\langle -k_4|\hat{O}_3\left(\frac{\sigma+\sigma'}{2}\right)|k_3\rangle + \ldots\Big] \\ = & |\eta_I| k_1 \delta_{k_1+k_2}\delta_{k_3+k_4} + \Big[\frac{i}{8}(k_1-k_2)^3\xi^2 f_3(k_3,k_4) \\ & \qquad + \frac{i}{32}(k_1-k_2)(k_3+k_4)^2\xi^2 f_3(k_3,k_4)\Big]\delta_{k_1+k_2+k_3+k_4} + \ldots \end{aligned} \tag{74}$$

where

$$\langle -k_4|O_3(\sigma)|k_3\rangle = e^{i(k_3+k_4)\sigma}\xi^2 f_3(k_3,k_4) \tag{75}$$

and ξ is a typical length scale in the microscopic theory. Notice that $O_3(\sigma)$ has a dimension (length)^2. $f_3(k_3,k_4)$ in (75) is a dimensionless function of order $O(1)$. For small k_i (*i.e.,* $k_i^2\xi^2 \ll 1$) the second term in (74) can be ignored comparing to the first term. Therefore (19) is valid in the small-momentum subspace.

In the following we would like to derive the commutators of the $SU(2)$ currents, (41). Following the above discussions for the $U(1)$ current, we can easily show that

$$\left[H, j^{a\pm}_{I,k}\right] = c_I k j^{a\pm}_{I,k} \tag{76}$$

$$\left[j^{a\pm}_{I,k}, j^{b\pm}_{J,k'}\right] = 0\ , \qquad I \neq J. \tag{77}$$

The current conservation $\partial_t j^{a0} + \partial_\sigma j^{a\sigma} = 0$ and (76) imply that

$$j^{a-}_I = 0 \tag{78}$$

in the small-momentum subspace.

Using the $SU(2)$ algebra

$$\int d\sigma \left[j^{a0}(\sigma), j^{b\alpha}_I(\sigma')\right] = \epsilon^{abc} j^{c\alpha}_I(\sigma') \tag{79}$$

we can show that (following a similar derivation leading to (65) and using (78))

$$\left[j^{a+}_{I,k=0}, j^{b+}_{I,k}\right] = \frac{1}{\sqrt{L}}\ \epsilon^{abc} j^{c+}_{I,k}. \tag{80}$$

In general the current commutator can be written as

$$\begin{aligned}\left[j^{a+}_I(\sigma), j^{b+}_I(\sigma')\right] = &-\frac{l_I}{2} i\delta'(\sigma-\sigma') + \delta(\sigma-\sigma')\hat{O}^{ab}_0\left(\frac{\sigma+\sigma'}{2}\right)\\ &+\delta'(\sigma-\sigma')\hat{O}^{ab}_1\left(\frac{\sigma+\sigma'}{2}\right) + \delta''(\sigma-\sigma')\hat{O}^{ab}_2\left(\frac{\sigma+\sigma'}{2}\right)\ldots\end{aligned} \tag{81}$$

where $\hat{O}^{ab}_i(\sigma)$ satisfy

$$\hat{O}^{ab}_i(\sigma) = -(-)^i \hat{O}^{ba}(\sigma) \tag{82}$$

The operators $\hat{O}^{ab}_i(\sigma)$ have zero vacuum expectation values as implied by (39). (79) implies that

$$\sqrt{\frac{\pi}{2}}\ \epsilon^{abc} j^{c+}_I(\sigma) = \hat{O}^{ab}_0(\sigma) - \frac{1}{2}\hat{O}^{ab\prime}_1(\sigma) - \frac{1}{4}\hat{O}^{ab\prime\prime}_2(\sigma) + \ldots \tag{83}$$

Using (82) we can break (83) into two equations

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\ \epsilon^{abc} j^{c+}_I(\sigma) &= \hat{O}^{ab}_0(\sigma) - \frac{1}{4}\hat{O}^{ab\prime\prime}_2(\sigma) - \frac{1}{16}\hat{O}^{ab\prime\prime\prime\prime}_4(\sigma) + \ldots\\ 0 &= -\frac{1}{2}\hat{O}^{ab\prime}_1(\sigma) - \frac{1}{8}\hat{O}^{ab\prime\prime\prime}_3(\sigma) + \ldots\end{aligned} \tag{84}$$

Because $\hat{O}_i^{ab}(\sigma)$ has a dimension $(\text{length})^{i-1}$, the matrix elements of $(\partial_\sigma)^i \hat{O}_i^{ab}(\sigma)$ in the small-momentum subspace contains a factor $(k\xi)^{i-1}$. Therefore in the small-momentum subspace the operators $\hat{O}_i^{ab}(\sigma)$, $i = 2, 3, \dots$ in (81) and (84) can be set to zero. In the previous calculation we also show that $\hat{O}_1^{ab}(\sigma)$ vanishes if $\hat{O}_1^{ab}(\sigma)$ satisfies (84) and $\langle 0|\hat{O}_1^{ab}(\sigma)|0\rangle = 0$. From (84) we find that $\hat{O}_0^{ab}(\sigma) = \frac{\pi}{\sqrt{2}}\epsilon^{abc} j_I^{c+}(\sigma)$. After dropping $\hat{O}_i(\sigma), i = 1, 2, \dots$ from (81), we find that (81) is equivalent to (41). In this way we show that (41) is valid in the small-momentum subspace.

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FIGURE CAPTIONS

Fig. 1: Electrons on the annulus form a QH state.

Fig. 2: The Feynmann diagram which contributes to the current-current correlation functions.

Fig. 3: The shaded region represents the energies and the momenta of the states created by the boundary current. (a) The low lying states created by the current have discrete velocities. (b) The low lying states have a continuous distribution of velocities.

Fig. 4: The energy spectrum (48) for (a) $c_I c_J > 0$ and (b) $c_I c_J < 0$. The dotted lines are unperturbed spectrums.