Quantum Field Theory of Many-body Systems
– from the Origin of Sound
to an Origin of Light and Fermions

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Abstract

For most of the last century, condensed matter physics has been dominated by band theory and Landau's symmetry breaking theory. In the last twenty years, however, there has been an emergence of a new paradigm associated with fractionalization, emergent gauge bosons and fermions, topological order, string-net condensation, and long range entanglements. These new physical concepts are so fundamental that they may even influence our understanding of the origin of light and electrons in the universe.

This book is a pedagogical and systematic introduction to the new concepts and quantum field theoretical methods in condensed matter physics. It discusses many basic notions in theoretical physics, which underlie physical phenomena in nature, including a notion that unifies light and electrons. Topics covered are dissipative quantum systems, boson condensation, symmetry breaking and gapless excitations, phase transitions, Fermi liquids, spin density wave states, Fermi and fractional statistics, quantum Hall effects, topological/quantum order, spin liquid and string-net condensation. Methods discussed include the path integral, Green's functions, mean-field theory, effective theory, renormalization group, bosonization in one- and higher dimensions, non-linear sigma-model, quantum gauge theory, dualities, projective construction, and exactly soluble models beyond one-dimension. This book is aimed at bringing students to the frontiers of research in condensed matter physics.

Key words: Condensed mater physics, many-body, quantum field theory, gauge theory, topological order, quantum matter, spin liquid, string-net condensation, quantum Hall effect, path integral, effective field theory

1. **Introduction**
   A quantitative change can lead to a qualitative change. The system with many degrees of freedom can demonstrate qualitatively new phenomena. In this chapter, we summarize the principle of emergence which states that the properties of material are mainly determined by the organizations (or the orders) of atoms in the material. The different organizations of microscopic degrees of freedom not only give rise to traditional symmetry breaking orders (such as crystal, magnets, superfluids, etc), they also give rise to new topological/quantum orders (such as fractional quantum Hall states, superconductors, etc). Those different orders lead to the rich properties of materials and the rich phenomena that we see every day. The collective excitations in the new topologically ordered states can even be gauge bosons and fermions that satisfy Coulombs law. So the principle of emergence in many-body system may explain the origin of light and electrons, as well as the beauty in law of physics.

   Key words: Emergence, order, symmetry breaking, effective theory, origin of light, origin of electron, topological order, quantum order, ether

2. **Path integral formulation of quantum mechanics**
   Path integral and various correlation functions at zero and non-zero temperatures are introduced to study interacting quantum systems. Semi classical approximation and instanton

effects are used to evaluate path integrals. The relation between correlation functions and physical measurements are discussed. The path integral method is then applied to a few simple systems, including a quantum system with friction and a quantum electric circuit.

Key words: Path integral, correlation function, Green function, propagator, semi classical approximation, instanton effect, dissipative quantum system, quantum electric circuit, Berry's phase

3. **Interacting boson systems**
   A quantum field theory for interacting boson systems is introduced. A mean-field theory is developed to study the superfluid phase. Then a path integral formulation is developed to re-derive the superfuid phase, which results in a low energy effective non-linear sigma model. A renormalization group approach is introduced to study the zero temperature quantum phase transition between superfuild and Mott insulator phase, and finite temperature phase transition between superfluid and normal phase. The physics and the importance of symmetry breaking in phase transitions and in protecting gapless excitations are discussed. The phenomenon of superfluidity and superconductivity is also discussed, where the coupling to $U(1)$ gauge field is introduced.

   Key words: Second quantization, boson operator, mean-field theory, non-linear sigma model, renormalization group, phase transition, superfluid, symmetry breaking, Goldstone mode, order parameter, Ginzburg-Landau theory, gauge coupling

4. **Free fermion systems**
   Quantum theory of many-fermion systems is introduced. Various correlation function and their related physical measurements are calculated for free fermion systems at both zero and non-zero temperatures. It is pointed out that many free fermion systems resemble critical points which are on verge to change into other qualitatively different phases. The topological properties of a filled band and its relation to quantized Hall conductance are also discussed.

   We also stress that a many-fermion system in two and higher dimension is not really a local quantum system. In one dimension, a many-fermion system can be viewed a local quantum system and the JordanCWigner transformation is introduced to transform a 1D many-fermion system to a 1D local many-boson system. We point out that the Fermi/Bose statistics is a dynamical property of the hopping Hamiltonian and comes from the statistical hopping algebra.

   Key words: Fermion operator, correlation functions, compressibility, spin susceptibility, diamagnetism, JordanCWigner transformation, statistical hopping algebra, quantized Hall conductance, topological band insulator, bosonization

5. **Interacting fermion systems**
   We first study the interaction effect of a Fermi sea on a single particle propagator and the related phenomenon of orthogonality catastrophe in X-ray spectrum. Next, We study a weakly interacting Fermi gas using Hartree-Fock approximation and introduce the notion of Fermi liquid. The concept and the appearance of quasiparticle in Fermi liquid are discussed. In addition to the standard quasiparticle description of Fermi liquid, we also introduced a hydrodynamic description of Fermi liquid, which can be viewed as higher dimensional bosonization.

   Then using the spin-density-wave state as an example, we discuss how interaction can cause instability of Fermi liquid and generate a symmetry breaking phase transition to a spin ordered phase. Several theoretical methods, such as mean-field theory, variatonal approach, and RPA method are introduced within the interacting fermion systems. The low energy O(3) non-linear sigma model for the spin ordered phase is derived, where we have paid a close attention to the possible topological terms in the non-linear sigma model. The effect of topological terms on the low energy dynamics of spin waves is discussed.

Key words: Orthogonality catastrophe, Fermi liquid theory, Hartree-Fock approximation, spin density wave, charge density wave, superconducting state, quasiparticle, topological term, spin wave, long range order

6. **Quantum gauge theories**
   In last a few chapters, we discussed many-boson theory and many-fermion theory because nature has many-boson systems and many-fermion systems. Do many-boson/many-fermion theories describe all the systems in nature? The answer is no. Nature also has photons. Although photons are bosons, they cannot be described by the local bosonic theory discussed in chapter 3. Photons are described by a non-local quantum theory which is called $U(1)$ gauge theory. In this chapter, we discuss $U(1)$ gauge theory as well as Z_2 gauge theory on lattice. We stress the non-localness in their Hilbert space. The gapless photonic, as well as the electric and the magnetic excitations are studied. We also discuss the confinement phase transition in those gauge theories, and a duality between $U(1)$ gauge theory and XY model in 1+2 dimensions.

   Key words: Gauge theory, gauge symmetry, photon, confinement, flux, vortex, charge, topological degeneracy, duality.

7. **Theory of quantum Hall states**
   Electrons on the interface of two semiconductors can form a new state of matter - fractional quantum Hall (FQH) state - under strong magnetic field. FQH states cannot be described by Landau symmetry breaking theory. So them shatter our long-hold believe that symmetry breaking theory describe all phases and phase transitions. As a result, a completely new theory is needed to describe FQH states, and this is the topic of this chapter.

   Many-electron system in strong magnetic field and resulting Landau level structure are studied. Laughlin's theory and the hierarchical theory for FQH effect are presented. We then derive the low energy effective Chern-Simons theory for FQH states and discuss the resulting fractional charge and fractional statistics, as well as the K-matrix classification of Abelian FQH states. The theory of chiral gapless edge states is also introduced, where experimental predictions can be made.

   Key words: Fractional quantum Hall effect, FQH effect, Laughlin state, hierarchical state, fractional charge, fractional statistics, Chern-Simons theory, edge state, conformal field theory, bosonization, K-matrix

8. **Topological and quantum orders**
   According to the principle of emergence, the properties of material are mainly determined by how the atoms are organized in the material. Such organization is formally called order. The vast variety of materials is a result of rich variety of orders that atoms can have. For a long time, we believe that all orders are described symmetry breaking. A comprehensive theory for phases and phase transitions is developed based on the symmetry breaking picture.

   The existence of FQH states (and superconducting states) indicates that there are new states of matter that cannot be described symmetry breaking. Completely new theory is needed to describe those new states of matter. In this chapter, we outline the theory of topological order and theory of quantum order for the new states of matter, such as FQH states. Many new concepts and new language, such as topology-dependent degeneracy, fractional statistics, edge states, etc, are introduced to describe new states of matter

Key words: Topological order, quantum order, symmetry breaking order, fractional charge, fractional statistics, Chern-Simons theory, topological degeneracy, topological field theory, phase transition

9. **Mean-field theory of spin liquids and quantum order**
   Topological order, as a generic phenomenon, not only appears in FQH systems, it can also appear in quantum spin systems. Quantum spin systems even allow the more general quantum order. In this chapter, we develop a mean-field theory for the topological/quantum order in strongly interacting quantum spin systems. The mean-field theory is based on the projective construction (or the slave-particle construction). We introduce the notion of projective symmetry group (PSG) to describe distinct phases that have exactly the same symmetry. PSG allows us to introduce the notion of quantum order which is more general then the notion of topological order. Using the mean-field theory, we calculate the phase diagram and phase transitions for quantum spin system that do not involve change of symmetry. In fact, many phases that we study do not break any symmetry and correspond to quantum spin liquids. The mean-field theory shows that those spin liquids can have some very exotic properties, such as fractionalization, spin-charge separation, emergent gauge bosons and fermions, interacting gapless exactions, etc. We find that, in addition to the symmetry breaking mechanism, PSG and quantum order is another way to produce and protect gapless excitations.

   Key words: Quantum order, projective symmetry group, PSG, spinon, spin liquid, projective construction, slave boson, slave particle, gauge theory, Z_2 spin liquid, $U(1)$ spin liquid, Z_2 topological order, Z_2 gauge theory

10. **String condensationan unification of light and fermions**
    We have discussed three types of theories, boson, fermion and gauge theories. Which theories describe the elementary particles in our universe? Surprisely, elementary particles are described the two more complicated and non-local theories, fermion theory and gauge theory.

    In last chapter and in this chapter, we demonstrate that we do not really need fermion and gauge theory. Boson theory may be able to explain everything, since both fermion theory and gauge theory can emerge from local bosonic models (or qbit models), if bosons (qbits) are organized to have a string-net condensed order.

    In this chapter, we study soluble quantum spin models to explain string-net condensation. We find that gauge bosons are simply collective excitations associated with string density wave (with the electric field corresponding to the string density). The fermons can appear as the ends of open strings. Thus the string-net order explains the emergence of gauge bosons and fermions. It unifies gauge interaction and Fermi statistics. We show that non-orientable strings give rise to Z_2 gauge theory while the orientable string give rise to $U(1)$ gauge theory.

    Key words: String-net condensation, emergence, gauge theory, fractionalization, exactly soluble model, charge, Maxwell equation, ether, unification of light and electrons.

Preface

The quantum theory of condensed matter (i.e. solids and liquids) has been dominated by two main themes. The first one is band theory and perturbation theory. It is loosely based on Landau's Fermi liquid theory. The second theme is Landau's symmetry-breaking theory and renormalization group theory. Condensed matter theory is a very successful theory. It allows us to understand the properties of almost all forms of matter. One triumph of the first theme is the theory of semiconductors, which lays the theoretical foundation for electronic devices that make recent technological advances possible. The second theme is just as important. It allows us to understand states of matter and phase transitions between them. It is the theoretical foundation behind liquid crystal displays, magnetic recording, etc.

As condensed matter theory has been so successful, one starts to get a feeling of completeness and a feeling of seeing the beginning of the end of condensed matter theory. However, this book tries to present a different picture. It advocates that what we have seen is just the end of the beginning. There is a whole new world ahead of us waiting to be explored.

A peek into the new world is offered by the discovery of the fraction quantum Hall effect (Tsui *et al.*, 1982). Another peek is offered by the discovery of high-T_c superconductors (Bednorz and Mueller, 1986). Both phenomena are completely beyond the two themes outlined above. In last twenty years, rapid and exciting developments in the fraction quantum Hall effect and in high-T_c superconductivity have resulted in many new ideas and new concepts. We are witnessing an emergence of a new theme in the many-body theory of condensed matter systems. This is an exciting time for condensed matter physics. The new paradigm may even have an impact on our understanding of fundamental questions of nature.

It is with this background that I have written this book.[1] The first half of this book covers the two old themes, which will be called traditional condensed matter theory.[2] The second part of this book offers a peek into the emerging new theme, which will be called modern condensed matter theory. The materials covered in the second part are very new. Some of them are new results that appeared only a few months ago. The theory is still developing rapidly.

After reading this book, I hope, instead of a feeling of completeness, readers will have a feeling of emptiness. After one-hundred years of condensed matter theory, which offers us so much, we still know so little about the richness of nature. However, instead of being disappointed, I hope that readers are excited by our incomplete understanding. It means that the interesting and exciting time of condensed matter theory is still ahead of us, rather than behind us. I also hope that readers will gain a feeling of confidence that there is no question that cannot be answered and no mystery

[1] When I started to write this book in 1996, I planned to cover some new and exciting developments in quantum many-body theory. At that time it was not clear if those new developments would become a new theme in condensed matter theory. At the moment, after some recent progress, I myself believe that a new theme is emerging in condensed matter theory. However, the theory is still in the early stages of its development. Only time will tell if we really do get a new theme or not.

[2] Some people may call the first theme traditional condensed matter theory and the second theme modern condensed matter theory.

that cannot be understood. Despite there being many mysteries which remain to be understood, we have understood many mysteries which initially seemed impossible to understand. We have understood some fundamental questions that, at the beginning, appeared to be too fundamental to even have an answer. The imagination of the human brain is also boundless.[3]

This book was developed when I taught the quantum many-body physics course between 1996 and 2002 at MIT. The book is intended for graduate students who are interested in modern theoretical physics. The first part (Chapters 2–5) covers traditional many-body physics, which includes path integrals, linear responses, the quantum theory of friction, mean-field theory for interacting bosons/fermions, symmetry breaking and long-range order, renormalization groups, orthogonality catastrophe, Fermi liquid theory, and nonlinear σ-models. The second part (Chapters 6–10) covers topics in modern many-body physics, which includes fractional quantum Hall theory, fractional statistics, current algebra and bosonization, quantum gauge theory, topological/quantum order, string-net condensation, emergent gauge-bosons/fermions, the mean-field theory of quantum spin liquids, and two- or three-dimensional exactly soluble models.

Most of the approaches used in this book are based on quantum field theory and path integrals. Low-energy effective theory plays a central role in many of our discussions. Even in the first part, I try to use more modern approaches to address some old problems. I also try to emphasize some more modern topics in traditional condensed matter physics. The second part covers very recent work. About half of it comes from research work performed in the last few years. Some of the second part is adapted from my research/review papers (while some research papers were adapted from parts of this book).

The book is written in a way so as to stress the physical pictures and to stress the development of thoughts and ideas. I do not seek to present the material in a neat and compact mathematical form. The calculations and the results are presented in a way which aims to expose their physical pictures. Instead of sweeping ugly assumptions under the rug, I try to expose them. I also stress the limitations of some common approaches by exposing (instead of hiding) the incorrect results obtained by those approaches.

Instead of covering many different systems and many different phenomena, only a few simple systems are covered in this book. Through those simple systems, we discuss a wide range of physical ideas, concepts, and methods in condensed matter theory. The texts in smaller font are remarks or more advanced topics, which can be omitted in the first reading.

Another feature of this book is that I tend to question and expose some basic ideas and pictures in many-body physics and, more generally, in theoretical physics, such as 'what are fermions?', 'what are gauge bosons?', the idea of phase transition and symmetry breaking, 'is an order always described by an order parameter?', etc. Here, we take nothing for granted. I hope that those discussions will encourage readers to look beyond the nice mathematical formulations that wrap many physical ideas, and to realize the ugliness and arbitrariness of some physical concepts.

As mathematical formalisms become more and more beautiful, it is increasingly easy to be trapped by the formalism and to become a 'slave' to the formalism. We used to be 'slaves' to Newton's laws when we regarded everything as a collection of particles. After the discovery of quantum theory,[4] we become 'slaves' to quantum field theory. At the moment, we want to use quantum field theory to explain everything and our education does not encourage us to look beyond quantum field theory.

However, to make revolutionary advances in physics, we cannot allow our imagination to be trapped by the formalism. We cannot allow the formalism to define the boundary of our imagina-

[3] I wonder which will come out as a 'winner', the richness of nature or the boundlessness of the human imagination.

[4] The concept of a classical particle breaks down in quantum theory. See a discussion in Section 2.2.

tion. The mathematical formalism is simply a tool or a language that allows us to describe and communicate our imagination. Sometimes, when you have a new idea or a new thought, you might find that you cannot say anything. Whatever you say is wrong because the proper mathematics or the proper language with which to describe the new idea or the new thought have yet to be invented. Indeed, really new physical ideas usually require a new mathematical formalism with which to describe them. This reminds me of a story about a tribe. The tribe only has four words for counting: one, two, three, and many-many. Imagine that a tribe member has an idea about two apples plus two apples and three apples plus three apples. He will have a hard time explaining his theory to other tribe members. This should be your feeling when you have a truly new idea. Although this book is entitled *Quantum field theory of many-body systems*, I hope that after reading the book the reader will see that quantum field theory is not everything. Nature's richness is not bounded by quantum field theory.

I would like to thank Margaret O'Meara for her proof-reading of many chapters of the book. I would also like to thank Anthony Zee, Michael Levin, Bas Overbosch, Ying Ran, Tiago Ribeiro, and Fei-Lin Wang for their comments and suggestions. Last, but not least, I would like to thank the copy-editor Dr. Julie Harris for her efforts in editing and polishing this book.

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Contents

Chapter 1

Introduction

1.1 More is different

- The collective excitations of a many-body system can be viewed as particles. However, the properties of those particles can be very different from the properties of the particles that form the many-body system.
- Guessing is better than deriving.
- Limits of classical computing.
- Our vacuum is just a special material.

A quantitative change can lead to a qualitative change. This philosophy is demonstrated over and over again in systems that contain many particles (or many degrees of freedom), such as solids and liquids. The physical principles that govern a system of a few particles can be very different from the physical principles that govern the collective motion of many-body systems. New physical concepts (such as the concepts of fermions and gauge bosons) and new physical laws and principles (such as the law of electromagnetism) can arise from the correlations of many particles (see Chapter 10).

Condensed matter physics is a branch of physics which studies systems of many particles in the 'condensed' (i.e. solid or liquid) states. The starting-point of current condensed matter theory is the Schrödinger equation that governs the motion of a number of particles (such as electrons and nuclei). The Schrödinger equation is mathematically complete. In principle, we can obtain all of the properties of any many-body system by solving the corresponding Schrödinger equation.

However, in practice, the required computing power is immense. In the 1980s, a workstation with 32 Mbyte RAM could solve a system of eleven interacting electrons. After twenty years the computing power has increased by 100-fold, which allows us to solve a system with merely two more electrons. The computing power required to solve a typical system of 10^{23} interacting electrons is beyond the imagination of the human brain. A classical computer made by all of the atoms in our universe would not be powerful enough to handle the problem.[1] Such an impossible computer could only solve the Schrödinger equation for merely about 100 particles.[2] We see that an generic interacting many-body system is an extremely complex system. Practically, it is impossible to deduce all of its exact properties from the Schrödinger equation. So, even if the Schrödinger

[1] It would not even have enough memory to store a single state vector of such a system.

[2] This raises a very interesting question—how does nature do its computation? How does nature figure out the state of 10^{23} particles one second later? It appears that the mathematics that we use is too inefficient. Nature does not do computations this way.

equation is the correct theory for condensed matter systems, it may not always be helpful for obtaining physical properties of an interacting many-body system.

Even if we do get the exact solution of a generic interacting many-body system, very often the result is so complicated that it is almost impossible to understand it in full detail. To appreciate the complexity of the result, let us consider a tiny interacting system of 200 electrons. The energy eigenvalues of the system are distributed in a range of about 200 eV. The system has at least 2^{200} energy levels. The level spacing is about $200\,\text{eV}/2^{200} = 10^{-60}\,\text{eV}$. Had we spent a time equal to the age of the universe in measuring the energy, then, due to the energy–time uncertainty relation, we could only achieve an energy resolution of order $10^{-33}\,\text{eV}$. We see that the exact result of the interacting many-body system can be so complicated that it is impossible to check its validity experimentally in full detail.[3] To really understand a system, we need to understand the connection and the relationship between different phenomena of a system. Very often, the Schrödinger equation does not directly provide such an understanding.

As we cannot generally directly use the Schrödinger equation to understand an interacting system, we have to start from the beginning when we are faced with a many-body system. We have to treat the many-body system as a black box, just as we treat our mysterious and unknown universe. We have to guess a low-energy effective theory that directly connects different experimental observations, instead of deducing it from the Schrödinger equation. We cannot assume that the theory that describes the low-energy excitations bears any resemblance to the theory that describes the underlying electrons and nuclei.

This line of thinking is very similar to that of high-energy physics. Indeed, the study of strongly-correlated many-body systems and the study of high-energy physics share deep-rooted similarities. In both cases, one tries to find theories that connect one observed experimental fact to another. (Actually, connecting one observed experimental fact to another is almost the definition of a physical theory.) One major difference is that in high-energy physics we only have one 'material' (our vacuum) to study, while in condensed matter physics there are many different materials which may contain new phenomena not present in our vacuum (such as fractional statistics, non-abelian statistics, and gauge theories with all kinds of gauge groups).

1.2 'Elementary' particles and physics laws are emergent phenomena

- Emergence—the first principle of many-body systems.
- Origin of 'elementary' particles.
- Origin of the 'beauty' of physics laws. (Why nature behaves reasonably.)

Historically, in our quest to understand nature, we have been misled by a fundamental (and incorrect) assumption that the vacuum is empty. We have (incorrectly) assumed that matter placed in a vacuum can always be divided into smaller parts. We have been dividing matter into smaller and smaller parts, trying to discover the smallest 'elementary' particles—the fundamental building block of our universe. We have been believing that the physics laws that govern the 'elementary' particles must be simple. The rich phenomena in nature come from these simple physics laws.

However, many-body systems present a very different picture. At high energies (or high temperatures) and short distances, the properties of the many-body system are controlled by the interaction between the atoms/molecules that form the system. The interaction can be very complicated

[3] As we cannot check the validity of the result obtained from the Schrödinger equation in full detail, our belief that the Schrödinger equation determines all of the properties of a many-body system is just a *faith*.

and specific. As we lower the temperature, depending on the form of the interaction between atoms, a crystal structure or a superfluid state is formed. In a crystal or a superfluid, the only low-energy excitations are collective motions of the atoms. Those excitations are the sound waves. In quantum theory, all of the waves correspond to particles, and the particle that corresponds to a sound wave is called a phonon.[4] Therefore, at low temperatures, a new 'world' governed by a new kind of particle—phonons—emerges. The world of phonons is a simple and 'beautiful' world, which is very different from the original system of atoms/molecules.

Let us explain what we mean by 'the world of phonons is simple and beautiful'. For simplicity, we will concentrate on a superfluid. Although the interaction between atoms in a gas can be complicated and specific, the properties of emergent phonons at low energies are simple and universal. For example, all of the phonons have an energy-independent velocity, regardless of the form of the interactions between the atoms. The phonons pass through each other with little interaction despite the strong interactions between the atoms. In addition to the phonons, the superfluid also has another excitation called rotons. The rotons can interact with each other by exchanging phonons, which leads to a dipolar interaction with a force proportional to $1/r^4$. We see that not only are the phonons emergent, but even the physics laws which govern the low-energy world of the phonons and rotons are emergent. The emergent physics laws (such as the law of the dipolar interaction and the law of non-interacting phonons) are simple and beautiful.

I regard the law of $1/r^4$ dipolar interaction to be beautiful because it is not $1/r^3$, or $1/r^{4.13}$, or one of billions of other choices. It is precisely $1/r^4$, and so it is fascinating to understand why it has to be $1/r^4$. Similarly, the $1/r^2$ Coulomb law is also beautiful and fascinating. We will explain the emergence of the law of dipolar interaction in superfluids in the first half of this book and the emergence of Coulomb's law in the second half of this book.

If our universe itself was a superfluid and the particles that form the superfluid were yet to be discovered, then we would only know about low-energy phonons. It would be very tempting to regard the phonon as an elementary particle and the $1/r^4$ dipolar interaction between the rotons as a fundamental law of nature. It is hard to imagine that those phonons and the law of the $1/r^4$ dipolar interaction come from the particles that are governed by a very different set of laws.

We see that in many-body systems the laws that govern the emergent low-energy collective excitations are simple, and those collective excitations behave like particles. If we want to draw a connection between a many-body system and our vacuum, then we should connect the low-energy collective excitations in the many-body system to the 'elementary' particles (such as the photon and the electron) in the vacuum. But, in the many-body system, the collective excitations are not elementary. When we examine them at short length scales, a complicated non-universal atomic/molecular system is revealed. Thus, in many-body systems we have collective excitations (also called quasiparticles) at low energies, and those collective excitations very often do not become the building blocks of the model at high energies and short distances. The theory at the atomic scale is usually complicated, specific, and unreasonable. The simplicity and the beauty of the physics laws that govern the collective excitations do not come from the simplicity of the atomic/molecular model, but from the fact that those laws have to allow the collective excitations to survive at low energies. A generic interaction between collective excitations may give those excitations a large energy gap, and those excitations will be unobservable at low energies. The interactions (or physics laws) that allow gapless (or almost gapless) collective excitations to exist must be very special—and 'beautiful'.

If we believe that our vacuum can be viewed as a special many-body material, then we have to conclude that there are no 'elementary' particles. All of the so-called 'elementary' particles in our vacuum are actually low-energy collective excitations and they may not be the building blocks

[4] A crystal has three kinds of phonons, while a superfluid has only one kind of phonon.

of the fundamental theory. The fundamental theory and its building blocks at high energies[5] and short distances are governed by a different set of physical laws. According to the point of view of emergence, those laws may be specific, non-universal, and complicated. The beautiful world and reasonable physical laws at low energies and long distances emerge as a result of a 'natural selection': the physical laws that govern the low-energy excitations should allow those excitations to exist at low energies. In a sense, the 'natural selection' explains why our world is reasonable.

Someone who knows both condensed matter physics and high-energy physics may object to the above picture because our vacuum appears to be very different from the solids and liquids that we know of. For example, our vacuum contains Dirac fermions (such as electrons and quarks) and gauge bosons (such as light), while solids and liquids seemingly do not contain these excitations. It appears that light and electrons are fundamental and cannot be emergent. So, to apply the picture of emergence in many-body systems to elementary particles, we have to address the following question: can gauge bosons and Dirac fermions emerge from a many-body system? Or, more interestingly, can gauge bosons and Dirac fermions emerge from a many-boson system?

The fundamental issue here is where do fermions and gauge bosons come from? What is the origin of light and fermions? Can light and fermions be an emergent phenomenon? We know that massless (or gapless) particles are very rare in nature. If they exist, then they must exist for a reason. But what is the reason behind the existence of the massless photons and nearly massless fermions (such as electrons)? (The electron mass is smaller than the natural scale—the Planck mass—by a factor of 10^{22} and can be regarded as zero for our purpose.) Can many-body systems provide an answer to the above questions?

In the next few sections we will discuss some basic notions in many-body systems. In particular, we will discuss the notion that leads to gapless excitations and the notion that leads to emergent gauge bosons and fermions from local bosonic models. We will see that massless photons and massless fermions can be emergent phenomena.

1.3 Corner-stones of condensed matter physics

- Landau's symmetry-breaking theory (plus the renormalization group theory) and Landau's Fermi liquid theory form the foundation of traditional condensed matter physics.

The traditional many-body theory is based on two corner-stones, namely Landau's Fermi liquid theory and Landau's symmetry-breaking theory (Landau, 1937; Ginzburg and Landau, 1950). The Fermi liquid theory is a perturbation theory around a particular type of ground state—the states obtained by filling single-particle energy levels. It describes metals, semiconductors, magnets, superconductors, and superfluids. Landau's symmetry-breaking theory points out that the reason that different phases are different is because they have different symmetries. A phase transition is simply a transition that changes the symmetry. Landau's symmetry-breaking theory describes almost all of the known phases, such as solid phases, ferromagnetic and anti-ferromagnetic phases, superfluid phases, etc., and all of the phase transitions between them.

Instead of the origin of light and fermions, let us first consider a simpler problem of the origin of phonons. Using Landau's symmetry-breaking theory, we can understand the origin of the gapless phonon. In Landau's symmetry-breaking theory, a phase can have gapless excitations if the ground state of the system has a special property called spontaneous breaking of the continuous symmetry (Nambu, 1960; Goldstone, 1961). Gapless phonons exist in a solid because the solid breaks the

[5] Here, by high energies we mean the energies of the order of the Planck scale $M_P = 1.2 \times 10^{19}$ GeV.

continuous translation symmetries. There are precisely three kinds of gapless phonons because the solid breaks three translation symmetries in the x, y, and z directions. Thus, we can say that the origin of gapless phonons is the translational symmetry breaking in solids.

It is quite interesting to see that our understanding of a gapless excitation—phonon—is rooted in our understanding of the phases of matter. Knowing light to be a massless excitation, one may perhaps wonder if light, just like a phonon, is also a Nambu–Goldstone mode from a broken symmetry. However, experiments tell us that a gauge boson, such as light, is really different from a Nambu–Goldstone mode in $3+1$ dimensions.

In the late 1970s, we felt that we understood, at least in principle, all of the physics about phases and phase transitions. In Landau's symmetry-breaking theory, if we start with a purely bosonic model, then the only way to get gapless excitations is via spontaneous breaking of a continuous symmetry, which will lead to gapless *scalar bosonic* excitations. It seems that there is no way to obtain gapless gauge bosons and gapless fermions from symmetry breaking. This may be the reason why people think that our vacuum (with massless gauge bosons and nearly-gapless fermions) is very different from bosonic many-body systems (which were believed to contain only gapless scalar bosonic collective excitations, such as phonons). It seems that there does not exist any order that gives rise to massless light and massless fermions. Due to this, we put light and fermions into a different category to phonons. We regard them as elementary and introduce them by hand into our theory of nature.

However, if we really believe that light and fermions, just like phonons, exist for a reason, then such a reason must be a certain order in our vacuum that protects their masslessness.[6] Now the question is what kind of order can give rise to light and fermions, and protect their masslessness? From this point of view, the very existence of light and fermions indicates that our understanding of the states of matter is incomplete. We should deepen and expand our understanding of the states of matter. There should be new states of matter that contain new kinds of orders. The new orders will produce light and fermions, and protect their masslessness.

1.4 Topological order and quantum order

- There is a new world beyond Landau's theories. The new world is rich and exciting.

Our understanding of this new kind of order starts at an unexpected place—fractional quantum Hall (FQH) systems. The FQH states discovered in 1982 (Tsui *et al.*, 1982; Laughlin, 1983) opened a new chapter in condensed matter physics. What is really new in FQH states is that we have lost the two corner-stones of the traditional many-body theory. Landau's Fermi liquid theory does not apply to quantum Hall systems due to the strong interactions and correlations in those systems. What is more striking is that FQH systems contain many *different* phases at zero temperature which have the *same* symmetry. Thus, those phases cannot be distinguished by symmetries and cannot be described by Landau's symmetry-breaking theory. We suddenly find that we have nothing in the traditional many-body theory that can be used to tackle the new problems. Thus, theoretical progress in the field of strongly-correlated systems requires the introduction of new mathematical techniques and physical concepts, which go beyond the Fermi liquid theory and Landau's symmetry-breaking principle.

[6] Here we have already assumed that light and fermions are not something that we place in an empty vacuum. Our vacuum is more like an 'ocean' which is not empty. Light and fermions are collective excitations that correspond to certain patterns of 'water' motion.

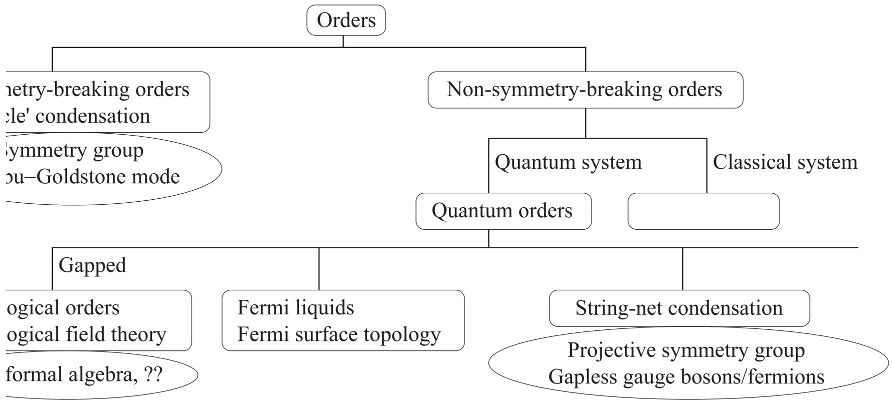

Figure 1.1: A classification of different orders in matter (and in a vacuum).

In the field of strongly-correlated systems, the developments in high-energy particle theory and in condensed matter theory really feed upon each other. We have seen a lot of field theory techniques, such as the nonlinear σ-model, gauge theory, bosonization, current algebra, etc., being introduced into the research of strongly-correlated systems and random systems. This results in a very rapid development of the field and new theories beyond the Fermi liquid theory and Landau's symmetry-breaking theory. This book is an attempt to cover some of these new developments in condensed matter theory.

One of the new developments is the introduction of quantum/topological order. As FQH states cannot be described by Landau's symmetry-breaking theory, it was proposed that FQH states contain a new kind of order—topological order (Wen, 1990, 1995). Topological order is new because it cannot be described by symmetry breaking, long-range correlation, or local order parameters. None of the usual tools that we used to characterize a phase apply to topological order. Despite this, topological order is not an empty concept because it can be characterized by a new set of tools, such as the number of degenerate ground states (Haldane and Rezayi, 1985), quasiparticle statistics (Arovas *et al.*, 1984), and edge states (Halperin, 1982; Wen, 1992).

It was shown that the ground-state degeneracy of a topologically-ordered state is robust against *any* perturbations (Wen and Niu, 1990). Thus, the ground-state degeneracy is a universal property that can be used to characterize a phase. The existence of topologically-degenerate ground states proves the existence of topological order. Topological degeneracy can also be used to perform fault-tolerant quantum computations (Kitaev, 2003).

The concept of topological order was recently generalized to quantum order (Wen, 2002c) to describe new kinds of orders in gapless quantum states. One way to understand quantum order is to see how it fits into a general classification scheme of orders (see Fig. 1.1). First, different orders can be divided into two classes: symmetry-breaking orders and non-symmetry-breaking orders. The symmetry-breaking orders can be described by a local order parameter and can be said to contain a condensation of point-like objects. The amplitude of the condensation corresponds to the order parameter. All of the symmetry-breaking orders can be understood in terms of Landau's symmetry-breaking theory. The non-symmetry-breaking orders cannot be described by symmetry breaking, nor by the related local order parameters and long-range correlations. Thus, they are a new kind of order. If a quantum system (a state at zero temperature) contains a non-symmetry-breaking order, then the system is said to contain a non-trivial quantum order. We see that a quantum order is simply a non-symmetry-breaking order in a quantum system.

Quantum orders can be further divided into many subclasses. If a quantum state is gapped, then the corresponding quantum order will be called the topological order. The low-energy effective theory of a topologically-ordered state will be a topological field theory (Witten, 1989). The second class of quantum orders appears in Fermi liquids (or free fermion systems). The different quantum orders in Fermi liquids are classified by the Fermi surface topology (Lifshitz, 1960). The third class of quantum orders arises from a condensation of nets of strings (or simply string-net condensation) (Wen, 2003a; Levin and Wen, 2003; Wen, 2003b). This class of quantum orders shares some similarities with the symmetry-breaking orders of 'particle' condensation.

We know that different symmetry-breaking orders can be classified by symmetry groups. Using group theory, we can classify all of the 230 crystal orders in three dimensions. The symmetry also produces and protects gapless collective excitations—the Nambu–Goldstone bosons—above the symmetry-breaking ground state. Similarly, different string-net condensations (and the corresponding quantum orders) can be classified by mathematical object called projective symmetry group (PSG) (Wen, 2002c). Using PSG, we can classify over 100 different two-dimensional spin liquids that all have the same symmetry. Just like the symmetry group, the PSG can also produce and protect gapless excitations. However, unlike the symmetry group, the PSG produces and protects gapless gauge bosons and fermions (Wen, 2002a,c; Wen and Zee, 2002). Because of this, we can say that light and massless fermions can have a unified origin; they can emerge from string-net condensations.

In light of the classification of the orders in Fig. 1.1, this book can be divided into two parts. The first part (Chapters 3–5) deals with the symmetry-breaking orders from 'particle' condensations. We develop the effective theory and study the physical properties of the gapless Nambu–Goldstone modes from the fluctuations of the order parameters. This part describes 'the origin of sound' and other Nambu–Goldstone modes. It also describes the origin of the law of the $1/r^4$ dipolar interaction between rotons in a superfluid. The second part (Chapters 7–10) deals with the quantum/topological orders and string-net condensations. Again, we develop the effective theory and study the physical properties of low-energy collective modes. However, in this case, the collective modes come from the fluctuations of condensed string-nets and give rise to gauge bosons and fermions. So, the second part provides 'an origin of light and electrons', as well as other gauge bosons and fermions. It also provides an origin of the $1/r^2$ Coulomb law (or, more generally, the law of electromagnetism).

1.5 Origin of light and fermions

- The string-net condensation provides an answer to the origin of light and fermions. It unifies gauge interactions and Fermi statistics.

We used to believe that, to have light and fermions in our theory, we have to introduce by hand a fundamental $U(1)$ gauge field and anti-commuting fermion fields, because at that time we did not know of any collective modes that behave like gauge bosons and fermions. However, due to the advances over the last twenty years, we now know how to construct *local bosonic systems* that have emergent *unconfined* gauge bosons and/or fermions (Foerster *et al.*, 1980; Kalmeyer and Laughlin, 1987; Wen *et al.*, 1989; Read and Sachdev, 1991; Wen, 1991a; Moessner and Sondhi, 2001; Motrunich and Senthil, 2002; Wen, 2002a; Kitaev, 2003; Levin and Wen, 2003). In particular, one can construct ugly bosonic spin models on a cubic lattice whose low-energy effective theory is the beautiful quantum electrodynamics (QED) and quantum chromodynamics (QCD) with emergent photons, electrons, quarks, and gluons (Wen, 2003b).

This raises the following issue: do light and fermions in nature come from a fundamental $U(1)$ gauge field and anti-commuting fields as in the $U(1) \times SU(2) \times SU(3)$ standard model, or do they

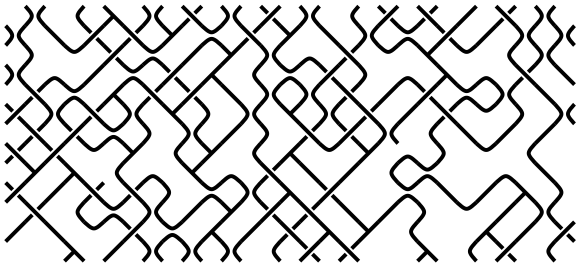

Figure 1.2: Our vacuum may be a state filled with a string-net. The fluctuations of the string-net give rise to gauge bosons. The ends of the strings correspond to electrons, quarks, etc.

come from a particular quantum order in our vacuum? Is Coulomb's law a fundamental law of nature or just an emergent phenomenon? Clearly, it is more natural to assume that light and fermions, as well as Coulomb's law, come from a quantum order in our vacuum. From the connections between string-net condensation, quantum order, and massless gauge/fermion excitations, we see that string-net condensation provides a way to unify light and fermions. It is very tempting to propose the following possible answers to the three fundamental questions about light and fermions.
What are light and fermions?
Light is the fluctuation of condensed string-nets (of arbitrary sizes). Fermions are ends of condensed strings.
Where do light and fermions come from?
Light and fermions come from the collective motions of string-nets that fill the space(see Fig. 1.2).

Why do light and fermions exist?
Light and fermions exist because our vacuum happens to have a property called string-net condensation.

Had our vacuum chosen to have ‘particle’ condensation, then there would be only Nambu–Goldstone bosons at low energies. Such a universe would be very boring. String-net condensation and the resulting light and fermions provide a much more interesting universe, at least interesting enough to support intelligent life to study the origin of light and fermions.

1.6 Novelty is more important than correctness

- The Dao that can be stated cannot be eternal Dao. The Name that can be named cannot be eternal Name. The Nameless is the origin of universe. The Named is the mother of all matter.[7]
- What can be stated cannot be novel. What cannot be stated cannot be correct.

In this introduction (and in some parts of this book), I hope to give the reader a sense of where we come from, where we stand, and where we are heading in theoretical condensed matter physics. I am not trying to summarize the generally accepted opinions here. Instead, I am trying to express my personal and purposely exaggerated opinions on many fundamental issues in condensed matter physics and high-energy physics. These opinions and pictures may not be correct, but I hope they are stimulating. From our experience of the history of physics, we can safely assume that none of the current physical theories are completely correct. (According to Lao Zi, the theory that can be written down cannot be the eternal theory, because it is limited by the mathematical symbols that we used to write down the theory.) The problem is to determine in which way the current theories are wrong and how to fix them. Here we need a lot of imagination and stimulation.

1.7 Remarks: evolution of the concept of elementary particles

- As time goes by, the status of elementary particles is downgraded from the building blocks of everything to merely collective modes of, possibly, a lowly bosonic model.

The Earth used to be regarded as the center of the universe. As times went by, its status was reduced to merely one of the billions of planets in the universe. It appears that the concept of elementary particles may have a similar fate.

At the beginning of human civilization, people realized that things can be divided into smaller and smaller parts. Chinese philosophers theorized that the division could be continued indefinitely, and hence that there were no elementary particles. Greek philosophers assumed that the division could not be continued indefinitely. As a result, there exist ultimate and indivisible particles—the building blocks of all matter. This may be the first concept of elementary particles. Those ultimate particles were called *atomos*. A significant amount of scientific research has been devoted to finding these *atomos*.

[7] These are the first four sentences of *Dao de jing* written by a Chinese philosopher Lao Zi over 2500 years ago. The above is a loose direct translation. Dao has meanings of ‘way’, ‘law’, ‘conduct’, etc. There are many very different translations of *Dao de jing*. It is interesting to search the Web and compare those different translations. The following is a translation in the context of this book. ‘The physical theory that can be formulated cannot be the final ultimate theory. The classification that can be implemented cannot classify everything. The unformulatable ultimate theory does exist and governs the creation of the universe. The formulated theories describe the matter we see every day.’

Around 1900, chemists discovered that all matter is formed from a few dozen different kinds of particles. People jumped the gun and named them atoms. After the discovery of the electron, people realized that elementary particles are smaller than atoms. Now, many people believe that photons, electrons, quarks, and a few other particles are elementary particles. Those particles are described by a field theory which is called the $U(1) \times SU(2) \times SU(3)$ standard model.

Although the $U(1) \times SU(2) \times SU(3)$ standard model is a very successful theory, now most high-energy physicists believe that it is not the ultimate theory of everything. The $U(1) \times SU(2) \times SU(3)$ standard model may be an effective theory that emerges from a deeper structure. The question is from which structure may the standard model emerge?

One proposal is the grand unified theories in which the $U(1) \times SU(2) \times SU(3)$ gauge group is promoted to $SU(5)$ or even bigger gauge groups (Georgi and Glashow, 1974). The grand unified theories group the particles in the $U(1) \times SU(2) \times SU(3)$ standard model into very nice and much simpler structures. However, I would like to remark that I do not regard the photon, electron, and other elementary particles to be emergent within the grand unified theories. In the grand unified theories, the gauge structure and the Fermi statistics were fundamental in the sense that the only way to have gauge bosons and fermions was to introduce vector gauge fields and anti-commuting fermion fields. Thus, to have the photon, electron, and other elementary particles, we had to introduce by hand the corresponding gauge fields and fermion fields. Therefore, the gauge bosons and fermions were added by hand into the grand unified theories; they did not emerge from a simpler structure.

The second proposal is the superstring theory (Green *et al.*, 1988; Polchinski, 1998). Certain superstring models can lead to the effective $U(1) \times SU(2) \times SU(3)$ standard model plus many additional (nearly) massless excitations. The gauge bosons and the graviton are emergent because the superstring theory itself contains no gauge fields. However, the Fermi statistics are not emergent. The electron and quarks come from the anti-commuting fermion fields on a $(1+1)$-dimensional world sheet. We see that, in the superstring theory, the gauge bosons and the gauge structures are not fundamental, but the Fermi statistics and the fermions are still fundamental.

Recently, people realized that there might be a third possibility—string-net condensation. Banks *et al.* (1977) and Foerster *et al.* (1980) first pointed out that light can emerge as low-energy collective modes of a local bosonic model. Levin and Wen (2003) pointed out that even three-dimensional fermions can emerge from a local bosonic model as the ends of condensed strings. Combining the two results, we find that the photon, electron, quark, and gluon (or, more precisely, the QED and the QCD part of the $U(1) \times SU(2) \times SU(3)$ standard model) can emerge from a local bosonic model (Wen, 2002a, 2003b) if the bosonic model has a string-net condensation. This proposal is attractive because the gauge bosons and fermions have a unified origin. In the string-net condensation picture, neither the gauge structure nor the Fermi statistics are fundamental; all of the elementary particles are emergent.

However, the third proposal also has a problem: we do not yet know how to produce the $SU(2)$ part of the standard model due to the chiral fermion problem. There are five deep mysteries in nature, namely, identical particles, Fermi statistics, gauge structure, chiral fermions, and gravity. The string-net condensation only provides an answer to the first three mysteries; there are two more to go.

where

$$\Psi_{a,\boldsymbol{k}}^{\top} = (\psi_{a,\boldsymbol{k}}, \psi_{a,\boldsymbol{k}+\boldsymbol{Q}_x}, \psi_{a,\boldsymbol{k}+\boldsymbol{Q}_y}, \psi_{a,\boldsymbol{k}+\boldsymbol{Q}_x+\boldsymbol{Q}_y}),$$
$$\boldsymbol{Q}_x = (\pi, 0, 0), \qquad \boldsymbol{Q}_y = (0, \pi, 0),$$
$$\Gamma(\boldsymbol{k}) = -8|\chi|N_f^{-1}(\sin(k_x)\Gamma_1 + \sin(k_y)\Gamma_2 + \sin(k_z)\Gamma_3) \tag{10.7.12}$$

and $\Gamma_1 = \tau^3 \otimes \tau^0$, $\Gamma_2 = \tau^1 \otimes \tau^3$, and $\Gamma_3 = \tau^1 \otimes \tau^1$. Here $\tau^{1,2,3}$ are the Pauli matrices and τ^0 is the 2×2 identity matrix. The momentum summation $\sum'_{\boldsymbol{k}}$ is over a range $k_x \in (-\pi/2, \pi/2)$, $k_y \in (-\pi/2, \pi/2)$, and $k_z \in (-\pi, \pi)$. As $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$, $i, j = 1, 2, 3$, we find that the fermions have the dispersion

$$E(\boldsymbol{k}) = \pm 8g|\chi|^3 N_f^{-1}\sqrt{\sin^2(k_x) + \sin^2(k_y) + \sin^2(k_z)}$$

We see that the dispersion has two nodes at $\boldsymbol{k} = 0$ and $\boldsymbol{k} = (0, 0, \pi)$. Thus, eqn (10.7.9) will give rise to $2N_f$ massless four-component Dirac fermions in the continuum limit.

After including the $U(1)$ gauge fluctuations, the massless Dirac fermions interact with the $U(1)$ gauge field as fermions with unit charge. Therefore, the total effective theory of our $SU(N_f)$ spin model is a QED with $2N_f$ families of Dirac fermions of unit charge. We will call these fermions artificial electrons. The continuum effective theory has the form

$$\mathcal{L} = \bar{\psi}_{I,a} D_0 \gamma^0 \psi_{I,a} + v_f \bar{\psi}_{I,a} D_i \gamma^i \psi_{I,a} + \frac{CN_f^2}{gl_0}\boldsymbol{E}^2 - (3l_0 g|\chi|^4)\boldsymbol{B}^2 + ... \tag{10.7.13}$$

where $I = 1, 2$, $D_0 = \partial_t + \mathrm{i}a_0$, $D_i = \partial_i + \mathrm{i}a_i|_{i=1,2,3}$, $v_f = 8l_0 g|\chi|^3/N_f^{-1}$, $\gamma_\mu|_{\mu=0,1,2,3}$ are 4×4 Dirac matrices, and $\bar{\psi}_{I,a} = \psi_{I,a}^\dagger \gamma^0$. Here $\psi_{1,a}$ and $\psi_{2,a}$ are Dirac fermion fields, which form a fundamental representation of $SU(N_f)$. We would like to point out that, although both the speed of the artificial light, c_a, and the speed of the artificial electrons, v_f, are of order $l_0 g/N_f$, the two speeds do not have to be the same in our model. Thus, Lorentz symmetry is not guaranteed.

Equation (10.7.13) describes the low-energy dynamics of the $SU(N_f)$ model in a quantum-ordered phase—the π-flux phase. The fermions and the gauge boson are massless and interact with each other. Here we would like to address an important question: after integrating out high-energy fermion and gauge fluctuations, do the fermions and the gauge boson remain massless? In general, the interaction between massless excitations will generate a mass term for them, unless the masslessness is protected by symmetry, or something else. For our $SU(N_f)$ model, the ground state breaks no symmetry. So we cannot use spontaneously broken symmetry to explain the massless excitations. The massless excitations are protected by the PSG that characterizes the quantum order (or string-net condensation) in the ground state (see Section 9.10 and Wen (2002a)).

10.7.4 Remarks: some historic remarks about gauge theory and Fermi statistics

- There are two ways to view a gauge field, namely as a geometric object of local phase invariance, or as a collective mode of a correlated system.
- The meaning of 'gauge'.
- Gauge fields and fermion fields do not imply gauge bosons and fermions as low-energy quasiparticles.

The first systematic gauge theory was Maxwell's theory for electromagnetism. Although the vector potential A_μ was introduced to express the electric field and the magnetic field, the meaning of A_μ was unclear.

The notion of a gauge field was introduced by Weyl in 1918, who also suggested that the vector potential A_μ is a gauge field. Weyl's idea is motivated by Einstein's theory of gravity and is an attempt to unify electromagnetism and gravity. In Einstein's general relativistic theory, the coordinate invariance leads to gravity. So Weyl thought that the invariance of another geometrical object may lead to electromagnetism. He proposed the scale invariance.

Consider a physical quantity that has a value f. We know that the numerical value f itself is meaningless unless we specify the unit. Let us use ω to denote the unit. The physical quantity is really given by $f\omega$. This is the relativity in scale. Now let us assume that the physical quantity is defined at every point in space (so we are considering a physical field). We would like to know how to compare the physical quantity at different points x^μ and $x^\mu + dx^\mu$. We cannot just compare the numerical values $f(x^\mu)$ and $f(x^\mu + dx^\mu)$ because the unit ω may be different at different points. For the nearby points x^μ and $x^\mu + dx^\mu$, the two units only differ by a factor close to 1. We can express such a factor as $1 + S_\mu dx^\mu$. The difference in the physical quantity at x^μ and $x^\mu + dx^\mu$ is not given by $f(x^\mu + dx^\mu) - f(x^\mu) = \partial_\mu f dx^\mu$, but by $f(x^\mu + dx^\mu)(1 + S_\mu dx^\mu) - f(x^\mu) = (\partial_\mu + S_\mu) f dx^\mu$. Weyl showed that the local scale invariance requires that only the curl of S_μ is physically meaningful, just like only the curl of A_μ is meaningful in Maxwell's theory. Thus, Weyl identified S_μ as the vector potential A_μ. Weyl called the local scale invariance 'Eich Invarianz', which was translated to 'gauge invariance'.

However, Weyl's idea is wrong and the vector potential A_μ cannot be identified as the 'gauge field' S_μ. On the other hand, Weyl was almost right. If we think of our physical field as the amplitude of a complex wave function[4] and the unit ω as a complex phase, i.e. $|\omega| = 1$, then the difference between the amplitudes at different points is given by $(\partial_\mu + \mathrm{i} S_\mu) f dx^\mu$, where the units at different points differ by a factor $(1 + \mathrm{i} S_\mu dx^\mu)$. It is such an S_μ that can be identified as the vector potential. So A_μ should really be called the 'phase field', and 'gauge invariance' should be called 'phase invariance'. However, the old name has stuck.

This part of history is an attempt to give the unphysical vector A_μ some physical (or geometrical) meaning. It views the vector potential as a connection of a fibre bundle. This picture is widely accepted. We now call the vector potential the gauge field, and Maxwell's theory is called gauge theory. However, this does not mean that we have to interpret the vector potential as a geometrical object from the local phase invariance. After all, the phase of a quantum wave function is unphysical.

There is another point of view about the gauge theory. Many thinkers in theoretical physics were not happy with the redundancy of the gauge potential A_μ. It was realized in the early 1970s that one could use gauge-invariant loop operators to characterize different phases of a gauge theory (Wegner, 1971; Wilson, 1974; Kogut and Susskind, 1975). Later, people found that one can formulate the entire gauge theory using closed strings (Banks *et al.*, 1977; Foerster, 1979; Gliozzi *et al.*, 1979; Mandelstam, 1979; Polyakov, 1979; Savit, 1980). These studies revealed the intimate relationship between gauge theories and closed-string theories—a point of view which is very different from the geometrical notion of vector potential.

In a related development in condensed matter physics, people found that gauge fields can emerge from a local bosonic model, if the bosonic model is in certain quantum phases. This phenomenon is also called the dynamical generation of gauge fields. The emergence of gauge fields from local bosonic models has a long and complicated history. The emergent $U(1)$ gauge field was introduced in the quantum-disordered phase of the $(1+1)$-dimensional CP^N model (D'Adda *et al.*, 1978; Witten, 1979). In condensed matter physics, the $U(1)$ gauge field has been found in the slave-boson approach to spin-liquid states of bosonic spin models on a square lattice (Affleck and Marston, 1988; Baskaran and Anderson, 1988). The slave-boson approach not only has a $U(1)$ gauge field, but it also has gapless fermion fields. However, due to the instanton effect and the resulting confinement of the $U(1)$ gauge field in $1+1$ and $1+2$ dimensions (Polyakov, 1975), none of the above gauge fields and gapless fermion fields lead to gauge bosons and gapless fermions that appear as low-energy physical quasiparticles. Even in the large-N limit where the instanton effect can be ignored, the marginal coupling between the $U(1)$ gauge field and the massless Dirac fermions in $2+1$ dimensions destroys the quasiparticle poles in the fermion and gauge propagators. This led to the opinion that the $U(1)$ gauge field and the gapless fermion fields are just an unphysical artifact of the 'unreliable' slave-boson approach. Thus, the key to finding emergent gauge bosons and emergent fermions is not to write down a Lagrangian that contains *gauge fields* and *Fermi fields*, but to show that gauge bosons and fermions actually appear in the physical low-energy spectrum. In fact, for any given physical system, we can always design a Lagrangian with a gauge field of arbitrary choice to describe that system. However, a gauge field in a Lagrangian may not give rise to a gauge boson that appears as a low-energy quasiparticle. Only when the dynamics of the gauge field are such that the gauge field is in the deconfined phase can the gauge boson appear as a low-energy quasiparticle. Thus, many researchers, after the initial findings of

[4]The notion of a complex wave function was introduced in 1925, seven years after Weyl's 'gauge theory'.

D'Adda *et al.* (1978), Witten (1979), Baskaran and Anderson (1988), and Affleck and Marston (1988), have been trying to find the deconfined phase of the gauge field.

In high-energy physics, a $(3+1)$-dimensional local bosonic model with emergent deconfined $U(1)$ gauge bosons was constructed by Foerster *et al.* (1980). It was suggested that light in nature may be emergent. In condensed matter physics, it was shown that, if we break the time-reversal symmetry in a two-dimensional spin-$1/2$ model, then the $U(1)$ gauge field from the slave-boson approach can be in a deconfined phase due to the appearance of the Chern–Simons term (Khveshchenko and Wiegmann, 1989; Wen *et al.*, 1989). The deconfined phase corresponds to a spin-liquid state of the spin-$1/2$ model (Kalmeyer and Laughlin, 1987), which is called the chiral spin liquid. A second deconfined phase was found by breaking the $U(1)$ gauge structure down to a Z_2 gauge structure. Such a phase contains a deconfined Z_2 gauge theory (Read and Sachdev, 1991; Wen, 1991a), and is called a Z_2 spin liquid (or a short-ranged RVB state). Both the chiral spin liquid and the Z_2 spin liquid have some amazing properties. The quasiparticle excitations carry spin-$1/2$ and correspond to *one-half* of a spin flip. These quasiparticles can also carry fractional statistics or Fermi statistics, despite our spin-$1/2$ model being a purely bosonic model. These condensed matter examples illustrate that both gauge fields and Fermi statistics can emerge from local bosonic models.

We would like to point out that the spin liquids are not the first example of emergent fermions from local bosonic models. The first example of emergent fermions, or, more generally, emergent anyons, is given by the FQH states. Although Arovas *et al.* (1984) only discussed how anyons can emerge from a fermion system in a magnetic field, the same argument can easily be generalized to show how fermions and anyons can emerge from a boson system in a magnetic field. Also, in 1987, in a study of resonating valence bound (RVB) states, emergent fermions (the spinons) were proposed in a nearest-neighbor dimer model on a square lattice (Kivelson *et al.*, 1987; Rokhsar and Kivelson, 1988; Read and Chakraborty, 1989). However, according to the deconfinement picture, the results by Kivelson *et al.* (1987) and Rokhsar and Kivelson (1988) are valid only when the ground state of the dimer model is in the Z_2 deconfined phase. It appears that the dimer liquid on a square lattice with only nearest-neighbor dimers is not a deconfined state (Rokhsar and Kivelson, 1988; Read and Chakraborty, 1989), and thus it is not clear if the nearest-neighbor dimer model on a square lattice (Rokhsar and Kivelson, 1988) has fermionic quasiparticles or not (Read and Chakraborty, 1989). However, on a triangular lattice, the dimer liquid is indeed a Z_2 deconfined state (Moessner and Sondhi, 2001). Therefore, the results of Kivelson *et al.* (1987) and Rokhsar and Kivelson (1988) are valid for the triangular-lattice dimer model, and fermionic quasiparticles do emerge in a dimer liquid on a triangular lattice.

All of the above models with emergent fermions are $(2+1)$-dimensional models, where the emergent fermions can be understood from binding flux to a charged particle (Arovas *et al.*, 1984). Recently, it was pointed out by Levin and Wen (2003) that the key to emergent fermions is a string structure. Fermions can generally appear as ends of open strings in any dimensions. The string picture allows the construction of a $(3+1)$-dimensional local bosonic model that has emergent fermions (Levin and Wen, 2003). Since both gauge bosons and fermions can emerge as a result of string-net condensation, we may say that string-net condensation provides a way to unify gauge bosons and fermions.

Generalizing the bosonic $SU(N)$ spin model on a two-dimensional square lattice (Affleck and Marston, 1988), both gapless deconfined $U(1)$ gauge bosons and gapless fermions were found to emerge from a bosonic $SU(N)$ spin model on a three-dimensional cubic lattice (Wen, 2002a). In $1+3$ dimensions, the two kinds of gapless excitations can be separated because they interact weakly at low energies. The $U(1)$ gauge bosons and gapless fermions behave in every way like photons and electrons. Thus, the bosonic $SU(N)$ spin model not only contains artificial light, but it also contains artificial electrons.

After about one hundred years of gauge theory and Fermi statistics, we are now facing the following questions. What is the origin of the gauge field—geometrical or dynamical? What is the origin of Fermi statistics—given or emergent? In this book, we favor the dynamical and emergent origin of gauge bosons and fermions. The gauge bosons and the Fermi statistics may just be collective phenomena of quantum many-boson systems, and nothing more.

Bibliography

Abrikosov, A. A., L. P. Gorkov, and I. E. Dzyaloshinski, 1975, *Method of Quantum Field Theory in Statistical Physics* (Dover, New York).

Affleck, I., T. Kennedy, E. H. Lieb, and H. Tasaki, 1987, Phys. Rev. Lett. **59**, 799.

Affleck, I., and J. B. Marston, 1988, Phys. Rev. B **37**, 3774.

Affleck, I., Z. Zou, T. Hsu, and P. W. Anderson, 1988, Phys. Rev. B **38**, 745.

Anderson, P. W., 1963, Phys. Rev. **130**, 439.

Appelquist, T., and D. Nash, 1990, Phys. Rev. Lett. **64**, 721.

Arovas, D., J. R. Schrieffer, and F. Wilczek, 1984, Phys. Rev. Lett. **53**, 722.

Arovas, D. P., and A. Auerbach, 1988, Phys. Rev. B **38**, 316.

Avron, J., R. Seiler, and B. Simon, 1983, Phys. Rev. Lett. **51**, 51.

Balents, L., M. P. A. Fisher, and S. M. Girvin, 2002, Phys. Rev. B **65**, 224412.

Balents, L., M. P. A. Fisher, and C. Nayak, 1998, Int. J. Mod. Phys. B **12**, 1033.

Banks, T., R. Myerson, and J. B. Kogut, 1977, Nucl. Phys. B **129**, 493.

Bardeen, J., L. N. Cooper, and J. R. Schrieffer, 1957, Phys. Rev. **108**, 1175.

Baskaran, G., and P. W. Anderson, 1988, Phys. Rev. B **37**, 580.

Baskaran, G., Z. Zou, and P. W. Anderson, 1987, Solid State Comm. **63**, 973.

Bednorz, J. G., and K. A. Mueller, 1986, Z. Phys. B **64**, 189.

Beenakker, C. W. J., 1990, Phys. Rev. Lett. **64**, 216.

Berry, M. V., 1984, Proc. R. Soc. Lond. **A392**, 45.

Blok, B., and X.-G. Wen, 1990a, Phys. Rev. B **42**, 8133.

Blok, B., and X.-G. Wen, 1990b, Phys. Rev. B **42**, 8145.

Blok, B., and X.-G. Wen, 1992, Nucl. Phys. B **374**, 615.

Brey, L., 1990, Phys. Rev. Lett. **65**, 903.

Buttiker, M., 1988, Phys. Rev. B **38**, 9375.

Caldeira, A. O., and A. J. Leggett, 1981, Phys. Rev. Lett. **46**, 211.

Cappelli, A., C. A. Trugenberger, and G. R. Zemba, 1993, Nucl. Phys. B **396**, 465.

Chaikin, P. M., and T. C. Lubensky, 2000, *Principles of Condensed Matter Physics* (Cambridge University Press).

Chang, A. M., L. N. Pfeiffer, and K. W. West, 1996, Phys. Rev. Lett. **77**, 2538.

Chen, W., M. P. A. Fisher, and Y.-S. Wu, 1993, Phys. Rev. B **48**, 13749.

Chen, Y. H., F. Wilczek, E. Witten, and B. Halperin, 1989, J. Mod. Phys. B **3**, 1001.

Coleman, S., 1975, Phys. Rev. D **11**, 2088.

Coleman, S., 1985, *Aspects of symmetry* (Pergamon, Cambridge).

Coleman, S., and E. Weinberg, 1973, Phys. Rev. D **7**, 1888.

D'Adda, A., P. D. Vecchia, and M. Lüscher, 1978, Nucl. Phys. B **146**, 63.

Dagotto, E., E. Fradkin, and A. Moreo, 1988, Phys. Rev. B **38**, 2926.

Dasgupta, C., and B. I. Halperin, 1981, Phys. Rev. Lett. **47**, 1556.

Eisenstein, J. P., T. J. Gramila, L. N. Pfeiffer, and K. W. West, 1991, Phys. Rev. B **44**, 6511.

Elitzur, S., G. Moore, A. Schwimmer, and N. Seiberg, 1989, Nucl. Phys. B **326**, 108.

Fertig, H., 1989, Phys. Rev. B **40**, 1087.

Fetter, A., C. Hanna, and R. Laughlin, 1989, Phys. Rev. B **39**, 9679.

Floreanini, R., and R. Jackiw, 1988, Phys. Rev. Lett. **59**, 1873.

Foerster, D., 1979, Physics Letters B **87**, 87.

Foerster, D., H. B. Nielsen, and M. Ninomiya, 1980, Phys. Lett. B **94**, 135.

Fradkin, E., and S. H. Shenker, 1979, Phys. Rev. D **19**, 3682.

Franz, M., and Z. Tesanovic, 2001, Phys. Rev. Lett. **87**, 257003.

Fröhlich, J., and T. Kerler, 1991, Nucl. Phys. B **354**, 369.

Fröhlich, J., and C. King, 1989, Comm. Math. Phys. **126**, 167.

Fröhlich, J., and U. M. Studer, 1993, Rev. of Mod. Phys. **65**, 733.

Georgi, H., and S. Glashow, 1974, Phys. Rev. Lett. **32**, 438.

Ginzburg, V. L., and L. D. Landau, 1950, Zh. Ekaper. Teoret. Fiz. **20**, 1064.

Girvin, S. M., and A. H. MacDonald, 1987, Phys. Rev. Lett. **58**, 1252.

Gliozzi, F., T. Regge, and M. A. Virasoro, 1979, Physics Letters B **81**, 178.

Goddard, P., and D. Olive, 1985, Workshop on Unified String Theories, eds. M. Green and D. Gross, (World Scientific, Singapore) , 214.

Goddard, P., and D. Olive, 1986, Inter. J. Mod. Phys. **1**, 303.

Goldstone, J., 1961, Nuovo Cimento **19**, 154.

Green, M. B., J. H. Schwarz, and E. Witten, 1988, *Superstring Theory* (Cambridge University Press).

Greiter, M., X.-G. Wen, and F. Wilczek, 1991, Phys. Rev. Lett. **66**, 3205.

Gross, D. J., J. A. Harvey, E. Martinec, and R. Rohm, 1985, Nucl. Phys. B **256**, 253.

Haldane, F., 1992, Helv. Phys. Acta. **65**, 152.

Haldane, F., and E. Rezayi, 1988a, Phys. Rev. Lett. **60**, E1886.

Haldane, F. D. M., 1983, Phys. Rev. Lett. **51**, 605.

Haldane, F. D. M., and E. H. Rezayi, 1985, Phys. Rev. B **31**, 2529.

Haldane, F. D. M., and E. H. Rezayi, 1988b, Phys. Rev. Lett. **60**, 956.

Halperin, B. I., 1982, Phys. Rev. B **25**, 2185.

Halperin, B. I., 1983, Helv. Phys. Acta **56**, 75.

Halperin, B. I., 1984, Phys. Rev. Lett. **52**, 1583.

Halperin, B. I., T. C. Lubensky, and S. K. Ma, 1974, Phys. Rev. Lett. **32**, 292.

Higgs, P. W., 1964, Phys. Rev. Lett. **12**, 132.

't Hooft, G., 1993, gr-qc/9310026 .

Houghton, A., and J. B. Marston, 1993, Phys. Rev. B **48**, 7790.

Ioffe, L. B., M. V. Feigel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, 2002, Nature **415**, 503.

Iso, S., D. Karabali, and B. Sakita, 1992, Phys. Lett. B **296**, 143.

Jain, J. K., and S. A. Kivelson, 1988a, Phys. Rev. B **37**, 4276.

Jain, J. K., and S. A. Kivelson, 1988b, Phys. Rev. Lett. **60**, 1542.

Jordan, P., and E. Wigner, 1928, Z. Phys. **47**, 631.

Kac, V. G., 1983, *Infinite dimensional Lie algebra* (Birkhauser, Boston).

Kalmeyer, V., and R. B. Laughlin, 1987, Phys. Rev. Lett. **59**, 2095.

Keldysh, L., 1965, Sov. Phys. JETP **20**, 1018.

Khveshchenko, D., and P. Wiegmann, 1989, Mod. Phys. Lett. **3**, 1383.

Kim, D. H., and P. A. Lee, 1999, Annals of Physics **272**, 130.

Kim, Y. B., P. A. Lee, and X.-G. Wen, 1995, Phys. Rev. B **52**, 17275.

Kitaev, A. Y., 2003, Ann. Phys. (N.Y.) **303**, 2.

Kivelson, S. A., D. S. Rokhsar, and J. P. Sethna, 1987, Phys. Rev. B **35**, 8865.

von Klitzing, K., G. Dorda, and M. Pepper, 1980, Phys. Rev. Lett. **45**, 494.

Kogut, J., and L. Susskind, 1975, Phys. Rev. D **11**, 395.

Kogut, J. B., 1979, Rev. Mod. Phys. **51**, 659.

Kosterlitz, J. M., and D. J. Thouless, 1973, J. Phys. C **6**, 1181.

Kotliar, G., 1988, Phys. Rev. B **37**, 3664.

Kotliar, G., and J. Liu, 1988, Phys. Rev. B **38**, 5142.

Landau, L. D., 1937, Phys. Z. Sowjetunion **11**, 26.

Landau, L. D., 1941, Zh. Eksp. Teor. Fiz. **11**, 592.

Landau, L. D., 1956, Sov. Phys. JETP **3**, 920.

Landau, L. D., 1959, Sov. Phys. JETP **8**, 70.

Landau, L. D., and E. M. Lifschitz, 1958, *Statistical Physics - Course of Theoretical Physics Vol 5* (Pergamon, London).

Laughlin, R. B., 1983, Phys. Rev. Lett. **50**, 1395.

Lee, P. A., and N. Nagaosa, 1992, Phys. Rev. B **45**, 5621.

Leinaas, J. M., and J. Myrheim, 1977, Il Nuovo Cimento **37B**, 1.

Levin, M., and X.-G. Wen, 2003, Phys. Rev. B **67**, 245316.

Lifshitz, I. M., 1960, Sov. Phys. JETP **11**, 1130.

Luther, A., 1979, Phys. Rev. B **19**, 320.

Luttinger, J. M., 1963, Journal of Mathematical Physics **4**, 1154.

Ma, S.-K., 1976, *Modern Theory of Critical Phenomena* (Benjamin/Cummings, Reading, MA).

MacDonald, A. H., 1990, Phys. Rev. Lett. **64**, 220.

MacDonald, A. H., P. M. Platzman, and G. S. Boebinger, 1990, Phys. Rev. Lett. **65**, 775.

MacDonald, A. H., and P. Streda, 1984, Phys. Rev. B **29**, 1616.

Mahan, G. D., 1990, *Many-Particle Physics* (Plenum, New York), 2nd edition.

Majumdar, C. K., and D. K. Ghosh, 1969, J. Math. Phys. **10**, 1388.

Mandelstam, S., 1979, Phys. Rev. D **19**, 2391.

Marston, J. B., and I. Affleck, 1989, Phys. Rev. B **39**, 11538.

Milliken, F. P., C. P. Umbach, and R. A. Webb, 1995, Solid State Comm. **97**, 309.

Moessner, R., and S. L. Sondhi, 2001, Phys. Rev. Lett. **86**, 1881.

Moore, G., and N. Read, 1991, Nucl. Phys. B **360**, 362.

Motrunich, O. I., 2003, Phys. Rev. B **67**, 115108.

Motrunich, O. I., and T. Senthil, 2002, Phys. Rev. Lett. **89**, 277004.

Mudry, C., and E. Fradkin, 1994, Phys. Rev. B **49**, 5200.

Murphy, S. Q., J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, and K. W. West, 1994, Phys. Rev. Lett. **72**, 728.

Nambu, Y., 1960, Phys. Rev. Lett. **4**, 380.

Negele, J. W., and H. Orland, 1998, *Quantum Many-Particle Systems* (Perseus Publishing).

Neto, A. H. C., and E. Fradkin, 1994, Phys. Rev. Lett. **72**, 1393.

Niu, Q., D. J. Thouless, and Y.-S. Wu, 1985, Phys. Rev. B **31**, 3372.

Onnes, H. K., 1911, Comm. Phys. Lab. Univ. Leiden, Nos 119 **120**, 122.

Polchinski, J., 1998, *String Theory* (Cambridge University Press).

Polyakov, A. M., 1975, Phys. Lett. B **59**, 82.

Polyakov, A. M., 1977, Nucl. Phys. B **120**, 429.

Polyakov, A. M., 1979, Phys. Lett. B **82**, 247.

Rammmer, J., and H. Smith, 1986, Rev. Mod. Phys. **58**, 323.

Ran, Y., and X.-G. Wen, 2003, unpublished .

Rantner, W., and X.-G. Wen, 2001, Phys. Rev. Lett. **86**, 3871.

Rantner, W., and X.-G. Wen, 2002, Phys. Rev. B **66**, 144501.

Read, N., 1989, Phys. Rev. Lett. **62**, 86.

Read, N., 1990, Phys. Rev. Lett. **65**, 1502.

Read, N., and B. Chakraborty, 1989, Phys. Rev. B **40**, 7133.

Read, N., and D. Green, 2000, Phys. Rev. B **61**, 10267.

Read, N., and S. Sachdev, 1989, Phys. Rev. Lett. **62**, 1694.

Read, N., and S. Sachdev, 1991, Phys. Rev. Lett. **66**, 1773.

Rokhsar, D. S., and S. A. Kivelson, 1988, Phys. Rev. Lett. **61**, 2376.

Sachdev, S., and K. Park, 2002, Annals of Physics (N.Y.) **298**, 58.

Savit, R., 1980, Rev. Mod. Phys. **52**, 453.

Schwinger, J., 1961, J. Math. Phys. **2**, 407.

Senthil, T., and M. P. A. Fisher, 2000, Phys. Rev. B **62**, 7850.

Senthil, T., and M. P. A. Fisher, 2001, Phys. Rev. Lett. **86**, 292.

Senthil, T., J. B. Marston, and M. P. A. Fisher, 1999, Phys. Rev. B **60**, 4245.

Senthil, T., and O. Motrunich, 2002, Phys. Rev. B **66**, 205104.

Streda, P., J. Kucera, and A. H. MacDonald, 1987, Phys. Rev. Lett. **59**, 1973.

Susskind, L., 1995, J. Math. Phys. **36**, 6377.

Thouless, D. J., M. Kohmoto, M. P. Nightingale, and M. den Nijs, 1982, Phys. Rev. Lett. **49**, 405.

Tomonaga, S., 1950, Prog. Theor. Phys. (Kyoto) **5**, 544.

Trugman, S. A., 1983, Phys. Rev. B **27**, 7539.

Tsui, D. C., H. L. Stormer, and A. C. Gossard, 1982, Phys. Rev. Lett. **48**, 1559.

Wegner, F., 1971, J. Math. Phys. **12**, 2259.

Wen, X.-G., 1990, Int. J. Mod. Phys. B **4**, 239.

Wen, X.-G., 1991a, Phys. Rev. B **44**, 2664.

Wen, X.-G., 1991b, Phys. Rev. Lett. **66**, 802.

Wen, X.-G., 1992, Int. J. Mod. Phys. B **6**, 1711.

Wen, X.-G., 1995, Advances in Physics **44**, 405.

Wen, X.-G., 2000, Phys. Rev. Lett. **84**, 3950.

Wen, X.-G., 2002a, Phys. Rev. Lett. **88**, 11602.

Wen, X.-G., 2002b, Physics Letters A **300**, 175.

Wen, X.-G., 2002c, Phys. Rev. B **65**, 165113.

Wen, X.-G., 2003a, Phys. Rev. B **68**, 115413.

Wen, X.-G., 2003b, Phys. Rev. D **68**, 065003.

Wen, X.-G., 2003c, Phys. Rev. Lett. **90**, 016803.

Wen, X.-G., and P. A. Lee, 1996, Phys. Rev. Lett. **76**, 503.

Wen, X.-G., and Q. Niu, 1990, Phys. Rev. B **41**, 9377.

Wen, X.-G., F. Wilczek, and A. Zee, 1989, Phys. Rev. B **39**, 11413.

Wen, X.-G., and Y.-S. Wu, 1993, Phys. Rev. Lett. **70**, 1501.

Wen, X.-G., Y.-S. Wu, and Y. Hatsugai, 1994, Nucl. Phys. B **422**, 476.

Wen, X.-G., and A. Zee, 1991, Phys. Rev. B **44**, 274.

Wen, X.-G., and A. Zee, 1992a, Phys. Rev. B **46**, 2290.

Wen, X.-G., and A. Zee, 1992b, Phys. Rev. Lett. **69**, 1811.

Wen, X.-G., and A. Zee, 1992c, Phys. Rev. Lett. **69**, 953.

Wen, X.-G., and A. Zee, 1992d, Phys. Rev. Lett. **69**, 3000.

Wen, X.-G., and A. Zee, 1993, Phys. Rev. B **47**, 2265.

Wen, X.-G., and A. Zee, 2002, Phys. Rev. B **66**, 235110.

Wilczek, F., 1982, Phys. Rev. Lett. **49**, 957.

Willett, R., J. P. Eisenstein, H. L. Strörmer, D. C. Tsui, A. C. Gossard, and J. H. English, 1987, Phys. Rev. Lett. **59**, 1776.

Wilson, K. G., 1974, Phys. Rev. D **10**, 2445.

Witten, E., 1979, Nucl. Phys. B **149**, 285.

Witten, E., 1989, Comm. Math. Phys. **121**, 351.

Yang, K., K. Moon, L. Zheng, A. H. MacDonald, S. M. Girvin, D. Yoshioka, and S.-C. Zhang, 1994, Phys. Rev. Lett. **72**, 732.

Zhang, S. C., T. H. Hansson, and S. Kivelson, 1989, Phys. Rev. Lett. **62**, 82.