

An Introduction to Quantum Order, String-net Condensation, and Emergence of Light and Fermions*

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We review some recent work on new states of matter. Those states contain a new kind of order – quantum order – because they cannot be described symmetry breaking. Some quantum orders are shown to be closely related to string-net condensations. Those quantum orders lead to an emergence of gauge bosons and fermions from pure bosonic models.

I. INTRODUCTION

A. Origin of light/fermion and new orders

The existences of light and fermions are two big mysteries in nature. The mysteries are so deep that the questions like, “What are light and fermions?”, “Where do light and fermions come from?”, “Why do light and fermions exist?”, are regarded by many people as philosophical or even religious questions.

To appreciate the physical significance those questions let us ask three simpler questions: “What are phonons?”, “Where do phonons come from?”, “Why do phonons exist?”. We know these three questions to be scientific questions and we know their answers. Phonons are vibrations of a crystal. Phonons come from a spontaneous translation symmetry breaking. Phonon exists because the translation-symmetry-breaking phase actually exists in nature. It is quite interesting to see that our understanding of a gapless excitation – phonon – is rooted in our understanding of the phases of matter as symmetry breaking states [1, 2].

However, our picture for massless photons and nearly massless fermions[70] (such as electrons and quarks) is quite different from our picture of gapless phonons. We regard photons and fermions as elementary particles – the building block of our universe.

But why should we regard photons and fermions as elementary particles? Why don’t we regard photons and fermions as emergent quasiparticles like phonons? We can view this question from several different points of view.

First point of view: Before late 1970’s, we felt that we understood, at least in principle, all the physics about phases and phase transitions based on Landau’s symmetry breaking theory [3, 4]. In such a theory, if we start with a bosonic model, the only way to get gapless excitations is via spontaneous breaking of a continuous

symmetry [1, 2], which will lead to gapless *scalar bosonic* excitations. It seems that there is no way to obtain gapless gauge bosons and fermions from symmetry breaking. This may be the reason why people think our vacuum (with massless gauge bosons and nearly-gapless fermions) is very different from bosonic many-body systems (which were believed to contain only gapless scalar bosonic collective excitations, such as phonons). It seems there does not exist any order that give rise to massless photons and nearly-massless fermions. This may be the reason why we regard photons and fermions as elementary particles and introduce them by hand into our theory of nature.

Second point of view: On the other hand, the resemblance between the photons and the phonons makes it odd to regard photons as elementary. To appreciate this point, let us imagine another universe which contains a massless excitation with three components. This massless excitation behaves in every way like the phonon in a crystal. We will not hesitate to declare that the vacuum in that universe is actually a crystal even when no one can see the particles that form the crystal. Our conviction of the existence of the crystal does not come from seeing the lattice structure, but from seeing the low energy collective modes of the crystal.

Third point of view: Now back to our universe. Are the massless photons and nearly massless fermions also collective modes of certain order in our vacuum. Not knowing what order can give rise to photons and fermions may not imply the photons and fermions to be elementary. More likely, it means that our understanding of order is incomplete. The very existence of light and fermions may indicate that our vacuum contain a new kind of order. The new order will produce light and fermions, and protect its masslessness.

Fourth point of view: If we had a material which is described by bosons (such as a spin system) and if we found that the low energy excitations in the material are gauge bosons and fermions, we would not hesitate to declare that the material contains a new kind of order beyond the symmetry breaking description. But so far, we have not find any material that contain emergent gauge bosons and emergent fermions. So we do not know if new order beyond the symmetry breaking exists or not. Other other hand, we may regard our vacuum as a special material. From this point of view, the light and the electrons in the vacuum provided an experimental evidence

*A more comprehensive description of topological/quantum orders can be found in *Quantum field theory of many-body systems – from the origin of sound to an origin of light and fermions*, Xiao-Gang Wen, Oxford Univ. Press, 2004.

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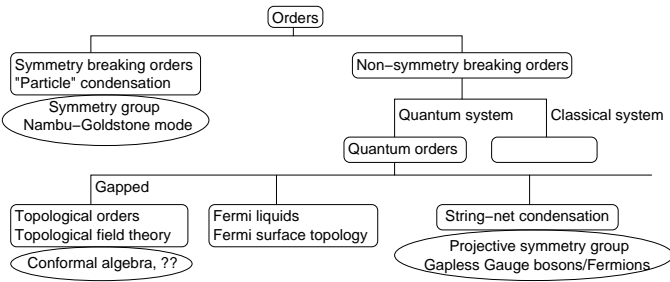


FIG. 1: A classification of different orders in matter (and in vacuum).

of the existence of new order.

B. Topological order and quantum order

Historically, our understanding of new order beyond symmetry breaking starts at an unexpected place — fractional quantum Hall (FQH) systems. The FQH states discovered in 1982 [5, 6] opened a new chapter of condensed matter physics. What is really new in FQH states is that FQH systems contain many *different* phases at zero temperature which have the *same* symmetry. Thus those phases cannot be distinguished by symmetries and cannot be described by Landau’s symmetry breaking theory.

Since FQH states cannot be described by Landau’s symmetry breaking theory, it was proposed that FQH states contain a new kind of order — topological order [7]. Topological order is new because it cannot be described by symmetry breaking, long range correlation, or local order parameters. None of the usual tools that we used to characterize a phase applies to topological order. Despite this, topological order is not an empty concept since it can be characterized by a new set of tools, such as the number of degenerate ground states [8, 9], the non-Abelian Berry’s phase under modular transformations [10], quasiparticle statistics [11], and edge states [12, 13]. Just like Ginzburg-Landau theory is the effective theory of symmetry breaking order, the topological field theory [14] is the effective theory of topological order [7].

It was shown that the ground state degeneracy of a topologically ordered state is robust against *any* perturbations [9]. Thus the ground state degeneracy is a universal property that can be used to characterize a phase. The existence of topologically degenerate ground states proves the existence of topological order. The topologically degenerate ground states were found to be useful in fault tolerant quantum computing [15].

The concept of topological order only applies to state with finite energy gap. It was recently generalized to quantum order [16] to describe new kind of orders in gapless quantum states. There are two general but vague understanding quantum orders.

In the first understanding, we assume that the order in a quantum state is encoded in the many-body ground state wave function. We believe that the symmetry of the ground state wave function cannot characterize all the possible orders in the many-body state. The extra structure in the ground state can be viewed as a pattern of quantum entanglement in the many-body state. From this point of view, we may say that quantum orders are patterns quantum entanglement in quantum many-body states.

The second way to understand quantum order is to see how it fits into a general classification scheme of orders (see Fig. 1). First, different orders can be divided into two classes: symmetry breaking orders and non-symmetry breaking orders. The symmetry breaking orders can be described by a local order parameter and can be said to contain a condensation of point-like objects. The amplitude of condensation corresponds to the order parameter. All the symmetry breaking orders can be understood in terms of Landau’s symmetry breaking theory. The non-symmetry breaking orders cannot be described by symmetry breaking, nor by the related local order parameters and long range correlations. Thus they are a new kind of orders. If a quantum system (a state at zero temperature) contains a non-symmetry breaking order, then the system is said to contain a non-trivial quantum order. We see that a quantum order is simply a non-symmetry breaking order in a quantum system.

Quantum orders can be further divided into many subclasses. If a quantum state is gapped, then the corresponding quantum order will be called topological order. The second class of quantum orders appear in Fermi liquids (or free fermion systems). The different quantum orders in Fermi liquids are classified by the Fermi surface topology [17]. We will discuss this class of quantum order briefly in section III. The third class of quantum orders arises from a condensation of nets of strings (string-nets) [18–24]. We will discuss it in sections IV and VI. This class of quantum orders shares some similarities with the symmetry breaking orders of “particle” condensation.

We know that different symmetry breaking orders can be classified by symmetry groups. Using group theory, we can classify all the 230 crystal orders in three dimensions. The phase transitions between different symmetry breaking orders are described by critical point with algebraic correlations. The symmetry also produces and protects gapless collective excitations — the Nambu-Goldstone bosons — above the symmetry breaking ground state. Similarly, different string-net condensations (and the corresponding quantum orders) can be classified by a mathematical object called projective symmetry group [16] (see subsection IV D). Using the projective symmetry group, we can classify over 100 different 2D spin liquids that all have the same symmetry. The phase transitions between different quantum orders are also described by critical points. Those phase transitions do not change any symmetry and cannot be described by order parameters associated with broken symmetries [25–29]. Just

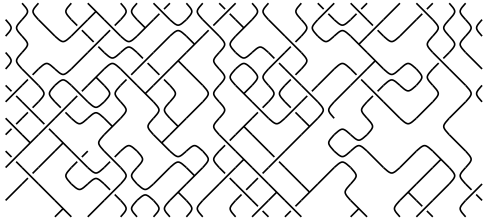


FIG. 2: Our vacuum may be a state filled with string-nets. The fluctuations of the string give rise to gauge bosons. The ends of the strings correspond to electrons, quarks, *etc.*

like the symmetry group, the projective symmetry group can also produce and protect gapless excitations. However, unlike the symmetry group, the projective symmetry group produces and protects gapless gauge bosons and fermions [16, 30, 31]. Because of this, we can say that light and massless fermions can have a unified origin. They can emerge from string-net condensations.

C. String-net picture of light and fermions

We used to believe that to have light and fermions in our theory, we have to introduce by hand a fundamental $U(1)$ gauge field and anti-commuting fermion fields, since at that time we did not know any collective modes that behave like gauge bosons and fermions. However, due to the advances of the last 20 years, we now know how to construct *local bosonic systems* that have emergent *unconfined* gauge bosons and/or fermions [15, 19, 30, 32–38]. In particular, one can construct ugly bosonic spin models on a cubic lattice whose low energy effective theory is the beautiful QED and QCD with emergent photons, electrons, quarks, and gluons [39].

This raises an issue: do light and fermions in nature come from a fundamental $U(1)$ gauge field and anti-commuting fields as in the $U(1) \times SU(2) \times SU(3)$ standard model or do they come from a particular quantum order in our vacuum? Is Coulomb’s law a fundamental law of nature or just an emergent phenomenon? Clearly it is more natural to assume light and fermions, as well as the Coulomb’s law, come from a quantum order in our vacuum. From the connections between string-net condensation, quantum order, and massless gauge/fermion excitations, it is very tempting to propose the following possible answers to the three fundamental questions about light and fermions:

What are light and fermions?

Light is the fluctuation of condensed strings (of arbitrary sizes) [21, 23, 37]. Fermions are ends of condensed strings [19].

Where do light and fermions come from?

Light and fermions come from the collective motions of nets of strings (or string-net) that fill our vacuum (see Fig. 2).

Why do light and fermions exist?

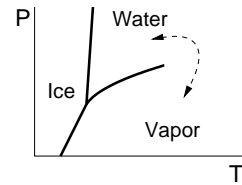


FIG. 3: The phase diagram of water.

Light and fermions exist because our vacuum happens to have a property called string-net condensation.

Had our vacuum chose to have “particle” condensation, there would be only Nambu-Goldstone bosons at low energies. Such a universe would be very boring. String condensation and the resulting light and fermions provide a much more interesting universe, at least interesting enough to support intelligent life to study the origin of light and fermions.

Our understanding of quantum/topological orders are based on many researches in three main areas: (1) the study of topological phases in condensed matter systems such as FQH systems [9, 40–42], quantum dimer models [32, 43–46], quantum spin models [10, 16, 33–36, 47–49], or even superconducting states [50, 51], (2) the study of lattice gauge theory [21–23, 52], and (3) the study of quantum computing by anyons [15, 53, 54]. In this paper, we will use some simple models to introduce the main points of topological/quantum order.

II. STATE OF MATTER AND CONCEPT OF ORDER

To start our journey to search new state of matter with emergent gauge bosons and fermions, we like to first discuss the concept of order and review the symmetry breaking description of order. With low temperature technology developed around 1900, physicists discovered many new states of matter (such as superconductors and superfluids). Those different states have different internal structures, which are called different kinds of orders. The precise definition of order involves phase transition. Two states of many-body systems have the same order if we can smoothly change one state into the other (by smoothly changing the Hamiltonian) without encounter a phase transition (*ie* without encounter a singularity in the free energy). If there is no way to change one state into the other without a phase transition, then the two states will have different orders. We note that our definition of order is a definition of equivalent class. Two states that can be connected without a phase transition are defined to be equivalent. The equivalent class defined this way is called the universality class. Two states with different orders can be also said as two states belong to different universality classes. According to our definition, water and ice have different orders while water and vapor have the same order. (See Fig. 3)

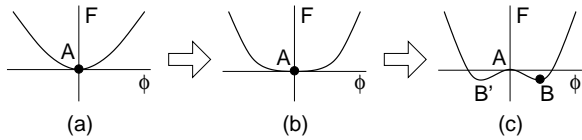


FIG. 4: In the presence of $\phi \rightarrow -\phi$ symmetry, switching of the minima can be continuous and causes a second-order phase transition. (a) The ground state is symmetric under the transformation $\phi \rightarrow -\phi$. (c) The ground state breaks the $\phi \rightarrow -\phi$ symmetry.

After discovering so many different kinds of orders, a general theory is needed to gain a deeper understanding of states of matter. In particular, we like to understand what make two orders really different so that we cannot change one order into the other without encounter a phase transition. It is a deep insight to connect the singularity in free energy to a symmetry breaking picture in Fig. 4. Based on the relation between orders and symmetries, Landau developed a general theory of orders and phase transitions [3, 55]. According to Landau's theory, the states in the same phase always have the same symmetry and the states in different phases always have different symmetries. So symmetry is a universal property that characterized different phases. Landau's theory is very successful. Using Landau's theory and the related group theory for symmetries, we can classify all the 230 different kinds of crystals that can exist in three dimensions. By determining how symmetry changes across a continuous phase transition, we can obtain the critical properties of the phase transition. The symmetry breaking also provides the origin of many gapless excitations, such as phonons, spin waves, *etc*, which determine the low energy properties of many systems [1, 2]. A lot of the properties of those excitations, including their gaplessness, are directly determined by the symmetry.

III. QUANTUM ORDERS AND QUANTUM TRANSITIONS IN FREE FERMION SYSTEMS

However, not all orders are described by symmetry. In fact, free fermion systems are the simplest systems with non-trivial quantum order. In this section, we will study the quantum order in a free fermion system to gain some intuitive understanding of quantum order.

To find quantum order, or to even define quantum order, we must find universal properties. The universal properties are the properties which do not change under any perturbations of the Hamiltonian that do not affect the symmetry. Once we find those universal properties, we can use them to group many-body wave functions into universal classes such that the wave functions in each class have the same universal properties. Hopefully those universal classes correspond to quantum phases in a phase diagram. To really show that those universal classes do correspond to quantum phases, we must show

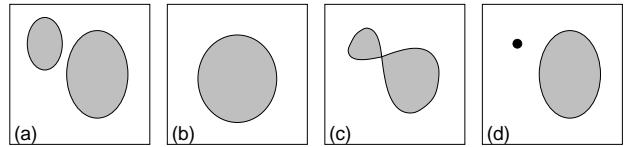


FIG. 5: Two sets of oriented Fermi surfaces in (a) and (b) represent two different quantum orders. The two possible transition points between the two quantum order (a) and (b) are described by the Fermi surfaces (c) and (d).

that as we deform the Hamiltonian to drive the ground state from one universality class to another, the ground state energy always has a singularity at the transition point. (For zero temperature quantum transition, the ground state energy play the role of free energy for finite temperature transition. A singularity in the ground state energy signal a quantum phase transition.)

We know that symmetry is a universal property. The order determined by such a universal property is our old friend – the symmetry-breaking order. So to show the existence of new quantum order, we must find universal properties that are different from symmetry.

Let us consider free fermion system with only the translation symmetry and the $U(1)$ symmetry from the fermion number conservation. The Hamiltonian has a form

$$H = \sum_{\langle ij \rangle} \left(c_i^\dagger t_{ij} c_j + h.c. \right) \quad (1)$$

with $t_{ij}^* = t_{ji}$. The ground state is obtained by filling every negative energy level with one fermion. In general, the system contains several pieces of Fermi surfaces.

We note that any small change of t_{ij} do not change the topology of the Fermi surfaces as long as the change do not break the translation symmetry and do not violate the fermion number conservation. So the the Fermi surface topology is a universal properties.

To show that the Fermi surface topology really defines quantum phases, we need to show that any change of the Fermi surface topology will lead to a singularity in the ground state energy. The Fermi surface topology can change in two ways as we continuously changing t_{ij} : (a) a Fermi surface shrinks to zero (Fig. 5d) and (b) two Fermi surfaces join (Fig. 5c).

When a Fermi surface is about to disappear in a D -dimensional system, the ground state energy density has a form

$$\rho_E = \int \frac{d^D \mathbf{k}}{(2\pi)^D} (\mathbf{k} \cdot M \cdot \mathbf{k} - \mu) \Theta(-\mathbf{k} \cdot M \cdot \mathbf{k} + \mu) + \dots$$

where the ... represents non-singular contribution and the symmetric matrix M is positive (or negative) definite. The integral in the above equation simply represents the total energy of the filled states enclosed by the small Fermi surface. The small Fermi surface is about to shrink to zero as μ pass zero. We find that

the ground state energy density has a singularity at $\mu = 0$: $\rho_E = c\mu^{(2+D)/2}\Theta(\mu) + \dots$, where $\Theta(x > 0) = 1$, $\Theta(x < 0) = 0$. When two Fermi surfaces are about to join, the singularity is still determined by the above equation, but now M has both negative and positive eigenvalues. The ground state energy density has a singularity $\rho_E = c\mu^{(2+D)/2}\Theta(\mu) + \dots$ when D is odd and $\rho_E = c\mu^{(2+D)/2}\log|\mu| + \dots$ when D is even.

We find that the ground state energy density has a singularity at $\mu = 0$ which is exactly the same place where the topology of the Fermi surfaces has a change [17]. Therefore the topology of the Fermi surface is a universal property that define a order. We note that the states with different Fermi surface topologies all have the same symmetry. Thus the quantum phase transition that change the the topology of the Fermi surface does not change any symmetry. Therefore the order defined by the Fermi surface topology is a new kind of order that cannot be characterized by symmetries. Such an order is an example of quantum order.

IV. QUANTUM ORDER IN BOSON/SPIN LIQUIDS

A. Quantum order and new universal properties

After realizing the existence of the quantum order in free fermion systems, we may expect quantum order to be a general phenomena. In this section we would like to study the existence of quantum order in interacting boson or spin systems. Instead of looking for universal properties, we would like to first look for boson/spin states that contain emergent gauge bosons and fermions. The emergence of gauge bosons and fermions indicate the appearance of new quantum order. Then we will study the universal properties which give a more systematic description of quantum orders.

We like to point out that a spin system is a special case of boson system since we can regard a site with a down spin as an empty site and a site with a up spin as a occupied site for bosons. In this section we will interchangeably use both the boson and the spin languages to describe the same system.

B. Projective construction

In the introduction, we argue that the existence of light and electron implies that our vacuum contains a non-trivial quantum order. However, we do not know to which system does the quantum order belong. Now we look for quantum order in spin systems. So we know our system. But we do not know what to look for, since we have no clue what does a quantum order look like at microscopic level. So instead of directly searching for quantum order, let us look for something slightly more

familiar: a spin liquid state that does not break any symmetry.

1. A mean-field theory of spin liquids

To be concrete, let us consider a spin-1/2 system on a square lattice

$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j. \quad (2)$$

In the conventional mean-field theory, we use the ground $|\Phi_{mean}^{\mathbf{m}_i}\rangle$ of a free spin Hamiltonian

$$H_{mean} = \sum_{\langle i \rangle} \mathbf{m}_i \mathbf{S}_i$$

to approximate the ground state of the interacting Hamiltonian H . The mean-field ground state described by $\mathbf{m}_i = \bar{\mathbf{m}}_i$, $|\Phi_{mean}^{\bar{\mathbf{m}}_i}\rangle$, is obtained by minimizing the average energy $\langle \Phi_{mean}^{\bar{\mathbf{m}}_i} | H | \Phi_{mean}^{\bar{\mathbf{m}}_i} \rangle$. However, no matter how we choose the mean-field ansatz \mathbf{m}_i , the mean-field ground state always break spin rotation symmetry and there is no way to obtain a spin liquid.

We have to use another approach to obtain a spin liquid [56, 57]. We start with a free fermion mean-field Hamiltonian that contains two fermion fields $\psi_i = \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix}$:

$$H_{mean} = \sum \psi_i^\dagger u_{ij} \psi_j \quad (3)$$

where u_{ij} are two by two complex matrices defined on the links $\langle ij \rangle$ that describe the hopping of the fermions. However, the mean-field ground state of H_{mean} , $|\Psi_{mean}^{u_{ij}}\rangle$, does not correspond to a spin state. But, we can obtain a spin state from the fermion state by projecting the fermion state $|\Psi_{mean}^{u_{ij}}\rangle$ into the subspace where every site has even numbers of fermion:

$$|\Phi_{spin}^{u_{ij}}\rangle = \mathcal{P} |\Psi_{mean}^{u_{ij}}\rangle$$

This is because there are only two states, on each site, that have even number of fermion. One is the empty site $|0\rangle$ which can be viewed as a spin-down state, and the other is the state with two fermions $\psi_{1i}^\dagger \psi_{2i}^\dagger |0\rangle$ which corresponds to the spin-up state.

Although it is not obvious, one can show [16] that iff u_{ij} satisfy

$$\text{Tr}(u_{ij}) = \text{imaginary}, \quad \text{Tr}(u_{ij}\tau^l) = \text{real}, \quad l = 1, 2, 3$$

where $\tau^{1,2,3}$ are the Pauli matrices, then $\Phi_{spin}^{u_{ij}}$ describes a spin rotation invariant state. Since the spin state is obtained through the projection \mathcal{P} , we will call the above construction projective construction. It is a special case of the slave-boson construction [56, 57] at zero doping.

We see that at least we can use the projective construction to construct a spin liquid state which does not break the spin rotation symmetry.

Through the projective construction, we introduced a label u_{ij} that labels a class of spin wave functions. (u_{ij} does not label all possible spin wave functions.) So we do not have to directly deal with the many-body functions of spin liquids which are very hard to visualize. We only need to deal with u_{ij} to understand the properties of the spin liquids.

2. The variational ground state

We may view u_{ij} as variational parameters. An approximate many-body ground state wave function $|\Phi_{spin}^{u_{ij}}\rangle$ can be obtained by minimizing the average energy $\langle \Phi_{spin}^{u_{ij}} | H | \Phi_{spin}^{u_{ij}} \rangle$ where H is the spin Hamiltonian.

Aside from its many variational parameters, there is no reason to expect the projective construction to give a good approximation of the ground state for a generic spin Hamiltonian. So there is no reason to trust any results obtained from projective construction. As we will see below that the projective construction leads to many unbelievable results, such as the emergence of gauge bosons, fermions, or even anyons from purely bosonic systems. So those amazing results may just be the artifacts of a unreliable approach.

However, for certain type of Hamiltonians, the projective construction leads to a good approximation of the ground state. In certain large N limits [33, 56], the fluctuations around the mean-field ansatz are weak, and the projective construction gives a good description of the ground state and the excitations. We can also construct special Hamiltonians where the projective construction leads to an exact ground state (and all the exact excited states. See section V). For those Hamiltonians, the projective construction does provide a good description of spin liquid states which cannot be provided by other conventional method. In the following, we will only consider those friendly Hamiltonians and trust the results of the projective construction.

I would like to mention that in practice, most of the unbelievable predictions from the projective construction turn out to be correct. For example, in the research of high T_c superconductors, both the d -wave superconducting state and the pseudo-gap metallic state was predicted by the projective construction prior to experimental observation [56, 58]. This may be the first time in the history of condensed matter physics that a truly new state of matter – the pseudo-gap metallic state – is predicted before the experimental observation [59].

3. Low energy excitations

In the conventional mean-field theory for spin ordered state, after we obtain the mean-field ground state

$|\Phi_{mean}^{\bar{\mathbf{m}}_i}\rangle$ that minimize the average energy, we can create collective excitations above the ground state through the fluctuation of the mean-field ansatz $\mathbf{m}_i = \bar{\mathbf{m}}_i + \delta\mathbf{m}_i$. Those collective excitations correspond to the spin wave excitations.

In the projective construction, we can create collective excitations in the exactly the same way. The collective excitations above the mean-field ground state $\Phi_{spin}^{\bar{u}_{ij}}$ correspond to the fluctuations of the mean-field ansatz $u_{ij} = \bar{u}_{ij} + \delta u_{ij}$. The physical spin wave function for such type of excitations is obtained via the projection of the deformed fermion state $|\Psi_{mean}^{\bar{u}_{ij} + \delta u_{ij}}\rangle$:

$$|\Phi_{spin}^{\delta u_{ij}}\rangle = \mathcal{P}|\Psi_{mean}^{\bar{u}_{ij} + \delta u_{ij}}\rangle$$

The ground state obtained from the projective construction also contains a second type of excitations. This type of excitations corresponds to fermion pair excitations. We start with the fermion ground state with a pair of particle-hole excitations $\psi_{ai}^\dagger \psi_{bj} |\Psi_{mean}^{\bar{u}_{ij}}\rangle$. After the projection, we obtain the corresponding physical spin state

$$|\Phi_{spin}^{(a,i,b,j)}\rangle = \psi_{ai}^\dagger \psi_{bj} |\Psi_{mean}^{\bar{u}_{ij}}\rangle$$

that describes a pair of fermions.

Clearly the fermions excitations interact with the collective modes δu_{ij} . The effective Lagrangian that describes the two types of excitations has a form $\mathcal{L}(\psi, \delta u_{ij})$. It appears that the spin liquid state obtained through the projective construction always contain fermionic excitations described by ψ . The emergent fermions will imply that the spin liquid state is a new state of matter and contain non-trivial quantum order.

However, the thing is not that easy. It turns out the collective fluctuations represent gauge fluctuations and the fermions carry gauge charges. Those fluctuations can mediate a confining interactions between the fermions. As a result, the spin liquid state may not contain any fermionic excitations and may not represent new state of matter.

To see that the fluctuations of u_{ij} represent gauge fluctuations, we note that the mean-field Hamiltonian H_{mean} is invariant under the following $SU(2)$ gauge transformation

$$\begin{aligned} \psi_i &\rightarrow W_i \psi_i, & W_i &\in SU(2) \\ u_{ij} &\rightarrow W_i u_{ij} W_j^\dagger \end{aligned} \quad (4)$$

So the two fermion ground states of the H_{mean} corresponding to two ansatz u_{ij} and u'_{ij} are related by an $SU(2)$ gauge transformation if $u'_{ij} = W_i u_{ij} W_j^\dagger$. Since the even-fermion states $|0\rangle$ and $\psi_{1i}^\dagger \psi_{2i}^\dagger |0\rangle$ are invariant under $SU(2)$ gauge transformation, the projected state $|\Phi_{spin}^{u_{ij}}\rangle = \mathcal{P}|\Psi_{mean}^{u_{ij}}\rangle$ is invariant under the $SU(2)$ gauge transformation:

$$|\Phi_{spin}^{u_{ij}}\rangle = |\Phi_{spin}^{W_i u_{ij} W_j^\dagger}\rangle$$

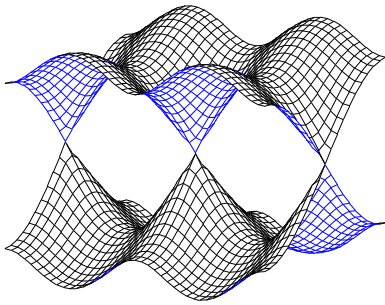


FIG. 6: The fermion dispersion for the ansatz Eq. (6).

As a result, u_{ij} is not a one-to-one label of the physical spin state, but a many-to-one label. The mean-field energy $E(u_{ij}) = \langle \Phi_{spin}^{u_{ij}} | H | \Phi_{spin}^{u_{ij}} \rangle$, as a function of real physical spin state $|\Phi_{spin}^{u_{ij}}\rangle$, is invariant under the $SU(2)$ gauge transformation $E(u_{ij}) = E(W_i u_{ij} W_j^\dagger)$. Similarly, the effective Lagrangian $\mathcal{L}(\psi, u_{ij})$ is also invariant under the $SU(2)$ gauge transformation:

$$\mathcal{L}(\psi, u_{ij}) = \mathcal{L}(W_i \psi, W_i u_{ij} W_j^\dagger)$$

The $SU(2)$ gauge invariance of the effective Lagrangian strongly affect the dynamics of u_{ij} fluctuations. It makes the u_{ij} fluctuations to behave like $SU(2)$ gauge fluctuations. If we write $u_{ij} = i\chi e^{ia_{ij}^l \tau^l}$, then a_{ij}^l play the role of the $SU(2)$ gauge potential on the lattice.

C. Deconfined phase and new state of matter

We have mentioned that to obtain a new state of matter from the projective construction, the gauge fluctuations u_{ij} must not mediate a confining interaction. One may say that the $SU(2)$ gauge fluctuations always mediate a confining interaction in 1+2D, so the projective construction can never produce a spin liquid with emergent fermions. Well again thing is not that simple. It turns out that whether the gauge fluctuations confine the fermions or not depend on the form of the mean-field ansatz \bar{u}_{ij} that minimize the average energy $\langle H \rangle$.

To understand how the ansatz \bar{u}_{ij} affect the dynamics of the gauge fluctuations, it is convenient to introduce the loop variable

$$P(C_i) = \bar{u}_{ij} \bar{u}_{jk} \dots \bar{u}_{li} \quad (5)$$

If we write $P(C_i)$ as $P(C_i) = \chi e^{i\Phi^l(C_i)\tau^l}$, then $\Phi^l \tau^l$ is the $SU(2)$ flux through the loop C_i : $\mathbf{i} \rightarrow \mathbf{j} \rightarrow \mathbf{k} \rightarrow \dots \rightarrow \mathbf{l} \rightarrow \mathbf{i}$ with base point \mathbf{i} . The $SU(2)$ flux correspond to the gauge field strength in the continuum limit.

1. $SU(2)$ spin liquid

If for a certain spin Hamiltonian H , \bar{u}_{ij} has a form

$$\begin{aligned} u_{i,i+\mathbf{x}} &= -i(-)^{i_y} \chi, \\ u_{i,i+\mathbf{y}} &= -i\chi, \end{aligned} \quad (6)$$

the $SU(2)$ flux through any loops is trivial. In this case the fluctuations of u_{ij} behave like the usual $SU(2)$ gauge fluctuations. The fermion dispersion is determined by mean-field Hamiltonian H_{mean} in Eq. (3): $E_{\mathbf{k}} = \pm 2\chi \sqrt{\sin^2 k_x + \sin^2 k_y}$. At low energies ($E_{\mathbf{k}} \sim 0$) the fermions have a linear dispersion $E \propto |\mathbf{k}|$ (see Fig. 6) and can be described by massless Dirac fermions in the continuum limit. So in the continuum limit, the low energy fluctuations are described by

$$\mathcal{L} = \sum_{\sigma=1}^N \bar{\psi}_\sigma (\partial_\mu - ia_\mu^l \tau^l) \gamma_\mu \psi_\sigma + \frac{1}{2g} \text{Tr} f_{\mu\nu}^l f^{l,\mu\nu} \quad (7)$$

where a_μ^l and $f_{\mu\nu}^l$, $l = 1, 2, 3$ the vector potential and the field strength of the $SU(2)$ gauge field.

It is interesting to see that for the spin liquid described by the ansatz Eq. (6), the low energy fluctuations is described by 1+2D QCD! The low energy physical properties of 1+2D QCD is complicated.

But the above construction can be easily generalize to higher dimensions. The low energy properties of 1+4D QCD is much easier to determine and it contains deconfined phase. As a result, the 1+4D spin liquid contains emergent massless fermions and emergent massless $SU(2)$ gauge bosons. Such a state represent a new state of matter and contains a non-trivial quantum order. We get what we are looking for (if we live in four dimensional space). Because of the low energy $SU(2)$ gauge fluctuations, we will call the ansatz Eq. (6) $SU(2)$ ansatz and the correspond spin liquid $SU(2)$ spin liquid.

2. $U(1)$ spin liquid

If the ansatz that minimizes the average energy has a form

$$\begin{aligned} u_{i,i+\mathbf{x}} &= -\tau^3 \chi - i(-)^i \Delta, \\ u_{i,i+\mathbf{y}} &= -\tau^3 \chi + i(-)^i \Delta, \end{aligned} \quad (8)$$

where $(-)^i \equiv (-)^{i_x + i_y}$, then the low energy u_{ij} fluctuations are actually described by $U(1)$ gauge fluctuations. This is because the above ansatz contains non-trivial $SU(2)$ flux: $P(C_i) \propto e^{i\Phi^3 \tau^3}$. Unlike the flux of $U(1)$ gauge field, the flux of $SU(2)$ gauge field is not invariant under the $SU(2)$ gauge transformations. Instead, the $SU(2)$ flux transforms like a Higgs field that carries non-zero $SU(2)$ charge. The non-zero $SU(2)$ flux has a similar effect as the condensation of a Higgs field, which can give gauge bosons a mass term via the Anderson-Higgs mechanism

[60, 61]. For our case, the $SU(2)$ flux in the τ^3 direction give the a_μ^1 and a_μ^2 components of the gauge field a mass, but a_μ^3 remains massless. So the $SU(2)$ flux break the $SU(2)$ gauge structure down to a $U(1)$ gauge structure [34]. This is why the spin liquid described by the ansatz Eq. (8) contains only massless $U(1)$ gauge fluctuations. The low energy fermions are still described by massless Dirac fermions. So the low energy effective theory of the spin liquid is a 1+2D QED

$$\mathcal{L} = \sum_{\sigma=1}^N \bar{\psi}_\sigma (\partial_\mu - ia_\mu^3 \tau^3) \gamma_\mu \psi_\sigma + \frac{1}{2g} f_{\mu\nu}^3 f^{3,\mu\nu} \quad (9)$$

where a_μ^3 and $f_{\mu\nu}^3$ are the vector potential and the field strength of the $U(1)$ gauge field. Again, the above construction can be easily generalize to higher dimensions. Now we do not need to go to 1+4 dimensions. In 1+3 dimensions, the spin liquid [30] already contains emergent massless fermions and emergent massless $U(1)$ gauge bosons since 1+3D QED is not confining. Such a 1+3D spin liquid represents another new state of matter and will be called $U(1)$ spin liquid.

The close resemblance of the low energy effective theory of the spin liquids and the QED/QCD in the standard model makes one really wonder: is this how the QED and QCD emerge to be the effective theory that describe our vacuum [39]?

Even in 1+2D, the $U(1)$ spin liquid was shown to be a stable phase [16, 62, 63] based a combined analysis of instanton [64] and projective symmetry [16] (see section IV D). The existence of the $U(1)$ spin liquid is a striking phenomenon since the gapless excitations interact down to zero energy, and yet remain to be gapless. The interaction is so strong that that are no free fermionic or bosonic quasiparticles at low energies. Since the $U(1)$ gauge bosons and the fermions are not well defined at any energy, the $U(1)$ spin liquid was more correctly called the algebraic spin liquid [16, 62].

Since there is no spontaneous broken symmetry to protect the above interacting gapless excitations, there should be a ‘‘principle’’ that prevents the gapless excitations from opening an energy gap and makes the algebraic spin liquids stable. Ref. [16] proposed that quantum order is such a principle. To support this idea, it was shown that just like the symmetry group of symmetry breaking order protects gapless Nambu-Goldstone modes, the projective symmetry group (see section IV D) of quantum order protects the interacting gapless excitations in the algebraic spin liquid. This result implies that the stabilities of algebraic spin liquids are protected by their projective symmetry groups. The existence of gapless excitations without symmetry breaking is a truly remarkable feature of quantum ordered states.

3. Z_2 spin liquid

For the ansatz [34]

$$\begin{aligned} u_{i,i+\mathbf{x}} &= u_{i,i+\mathbf{y}} = -\chi\tau^3, \\ u_{i,i+\mathbf{x}+\mathbf{y}} &= \eta\tau^1 + \lambda\tau^2, \\ u_{i,i-\mathbf{x}+\mathbf{y}} &= \eta\tau^1 - \lambda\tau^2, \end{aligned} \quad (10)$$

the $SU(2)$ flux $\Phi^l(C_i)\tau^l$ for different loops points in different directions in the (τ^1, τ^2, τ^3) space. The non-collinear $SU(2)$ flux break the $SU(2)$ gauge structure down to a Z_2 gauge structure. Since the Z_2 gauge fluctuations only mediate a short ranged interaction, the fermions are not confined even in 1+2D. The spin liquid obtained from the ansatz Eq. (10) is a new state of matter that has emergent fermions and Z_2 gauge theory. Non-trivial quantum order can appear in two dimensional space. The spin liquid will be called Z_2 spin liquid. Such a spin liquid corresponds to the short-ranged Resonating Valence Bound state proposed in Ref. [43, 65].

4. Summary

The projective construction is a powerful way construct states that represent new state of matter. Those states have emergent fermions and gauge bosons, and thus contain a new kind of order that cannot be described by symmetry. The new order is called quantum orders. Certainly, not all states obtained via the projection construction contain non-trivial quantum orders. But many of them do. The projective construction not only can produce states with emergent QED and QCD, it can also produce states with fractional statistics (including non-Abelian statistics) [36, 66, 67].

D. Quantum order and projective symmetry group

We know that different symmetry-breaking orders can be systematically characterize by different symmetry group. The group theory description allows us to classify 230 different 3D crystals. Knowing the existence of new quantum order in spin liquids, we would like to ask what mathematical object that we can use to systematically describe different quantum orders? In this subsection, we are going to introduce a mathematical object – projective symmetry group and show that the projective symmetry group can (partially) characterize different quantum orders.

1. The difficulty of seeing quantum order

In subsection IV A, we argue that to describe or to even define quantum order, we must find universal properties

that are different from the symmetry (such as the topology of the Fermi surfaces discussed in the section III). However, it is very difficult to find new universal properties of generic many-body wave functions. Let us consider the free fermion systems that we discussed before to gain some intuitive understanding of the difficulty. We know that a free fermion ground state is described by an anti-symmetric wave function of N variables. The anti-symmetric function has a form of Slater determinant: $\Psi(x_1, \dots, x_N) = \det(M)$ where the matrix elements of M is given by $M_{mn} = \psi_n(x_m)$ and ψ_n are single-fermion wave functions. The first step to find quantum orders in free fermion systems is to find a reasonable way to group the Slater-determinant wave functions into classes. This is very difficult to do if we only know the real space many-body function $\Psi(x_1, \dots, x_N)$. However, if we use Fourier transformation to transform the real-space wave function to momentum-space wave function, then we can group different wave functions into classes according to their Fermi surface topologies. This leads to our understanding of quantum orders in free fermions systems (see section III). The Fermi surface topology is the quantum number that allows us characterize different quantum phases of free fermions. Here we would like to stress that without the Fourier transformation, it is very difficult to see Fermi surface topologies from the real space many-body function $\Psi(x_1, \dots, x_N)$.

For the boson/spin systems, what is missing here is the corresponding ‘‘Fourier’’ transformation. Just like the topology of Fermi surface, it is very difficult to see universal properties (if any) directly from the real space wave function. At moment there are two ways to understand the quantum order in boson/spin systems. The first one is through the projective symmetry group which will be discussed below. The second one is through string-net condensation which will be discuss in section VI. Both the projective symmetry group and the string-net condensation play the role of the Fourier transformation in the free fermion system. They allow us the extract the universal properties from the very complicated many-body wave functions.

2. Symmetry of the spin liquids

To motivate the projective symmetry group, let us first consider the symmetry of the spin liquid states obtained from the $SU(2)$, $U(1)$ and Z_2 ansatz Eq. (6), Eq. (8), and Eq. (10). At first sight, those spin liquids appear not to have all the symmetries. For example, the $U(1)$ ansatz Eq. (8) is not invariant under translation in the x -direction.

However, those ansatz do describe spin states that have all the symmetries of square lattice, namely the two translation symmetries $T_x: (i_x, i_y) \rightarrow (i_x + 1, i_y)$ and $T_y: (i_x, i_y) \rightarrow (i_x, i_y + 1)$, and three parity symmetries, $P_x: (i_x, i_y) \rightarrow (-i_x, i_y)$, $P_y: (i_x, i_y) \rightarrow (i_x, -i_y)$, and $P_{xy}: (i_x, i_y) \rightarrow (i_y, i_x)$. This is because the ansatz u_{ij}

is a many-to-one label of the physical spin state. The non-invariance of the ansatz does not imply the non-invariance of the corresponding physical spin state after the projection. We only require the mean-field ansatz to be invariant up to a $SU(2)$ gauge transformation in order for the projected physical spin state to have a symmetry. For example, a T_x translation transformation changes the $U(1)$ ansatz Eq. (8) to

$$\begin{aligned} U_{i, i+x} &= -\tau^3 \chi + i(-)^i \Delta, \\ U_{i, i+y} &= -\tau^3 \chi - i(-)^i \Delta, \end{aligned}$$

The translated ansatz can be transformed into the original ansatz via a $SU(2)$ gauge transformation $W_i = (-)^i i \tau^1$. Therefore, after the projection, the ansatz Eq. (8) describes a T_x translation symmetric spin state.

Using the similar consideration, one can show that the $SU(2)$, $U(1)$, and Z_2 ansatz are invariant under translation $T_{x,y}$ and parity $P_{x,y,xy}$ symmetry transformations followed by corresponding $SU(2)$ gauge transformations G_{T_x, T_y} and $G_{P_x, P_y, P_{xy}}$ respectively. Thus the three ansatz all describe symmetric spin liquids. In the following, we list the corresponding gauge transformations G_{T_x, T_y} and $G_{P_x, P_y, P_{xy}}$ for the three ansatz: for the $SU(2)$ ansatz Eq. (6):

$$\begin{aligned} G_{T_x}(\mathbf{i}) &= (-)^{i_x} G_{T_y}(\mathbf{i}) = \tau^0, & G_{P_{xy}}(\mathbf{i}) &= (-)^{i_x i_y} \tau^0, \\ (-)^{i_x} G_{P_x}(\mathbf{i}) &= (-)^{i_y} G_{P_y}(\mathbf{i}) = \tau^0, & G_0(\mathbf{i}) &= e^{i\theta^1 \tau^1} \end{aligned} \quad (11)$$

for the $U(1)$ ansatz Eq. (8):

$$\begin{aligned} G_{T_x}(\mathbf{i}) &= G_{T_y}(\mathbf{i}) = i(-)^i \tau^1, & G_{P_{xy}}(\mathbf{i}) &= i(-)^i \tau^1, \\ G_{P_x}(\mathbf{i}) &= G_{P_y}(\mathbf{i}) = \tau^0, & G_0(\mathbf{i}) &= e^{i\theta^3} \end{aligned} \quad (12)$$

for the Z_2 ansatz Eq. (10):

$$\begin{aligned} G_{T_x}(\mathbf{i}) &= G_{T_y}(\mathbf{i}) = i\tau^0, & G_{P_{xy}}(\mathbf{i}) &= \tau^0, \\ G_{P_x}(\mathbf{i}) &= G_{P_y}(\mathbf{i}) = (-)^i \tau^1, & G_0(\mathbf{i}) &= -\tau^0 \end{aligned} \quad (13)$$

In the above we also list the pure gauge transformation $G_0(\mathbf{i})$ that leave the ansatz invariant: $\bar{u}_{ij} = G_0(\mathbf{i}) \bar{u}_{ij} G_0^\dagger(\mathbf{j})$.

3. Definition of PSG

The $SU(2)$, $U(1)$ and Z_2 ansatz after the projection, give rise to three spin liquid states. The three states have the exactly the same symmetry. The question here is whether the three spin liquids belong to the same phase or not. According to Landau’s symmetry breaking theory, two states with the same symmetry belong to the same phase. However, we now know that Landau’s symmetry breaking theory does not describe all the phases. It is possible that the three spin liquids contain different orders that cannot be characterized by symmetries. The

issue here is to find a new set of quantum numbers that characterize the new orders.

To find a new set of universal quantum numbers that distinguish the three spin liquids, we note that although the three spin liquids have the same symmetry, their ansatz are invariant under the symmetry translations followed by different gauge transformations (see Eq. (11), Eq. (12), and Eq. (13)). So the invariant group of three ansatz are different. We can use the invariant group of the three ansatz to characterize the new order in the spin liquid. In a sense, the invariant group define a new order – quantum order.

The invariant group of an ansatz is formed by all the combined symmetry transformations and the gauge transformations that leave the ansatz invariant. Those combined transformations from a group. Such a group is called the Projective Symmetry Group (PSG). The combined transformations ($G_{T_x}T_x, G_{T_y}T_y, G_{P_x}P_x, G_{P_y}P_y, G_{P_{xy}}P_{xy}$) and G_0 in Eq. (11), Eq. (12), and Eq. (13) generate the three PSG's for the three ansatz Eq. (6), Eq. (8), and Eq. (10).

4. Properties of PSG

To understand the properties of the PSG, we would like to point out that a PSG contains a special subgroup, which will be called the invariant gauge group (IGG). An IGG is formed by pure gauge transformations that leave the ansatz unchanged

$$IGG \equiv \{G_0 | u_{ij} = G_0(\mathbf{i})u_{ij}G_0^\dagger(\mathbf{j})\}$$

For the ansatz Eq. (6), Eq. (8), and Eq. (10), the IGG's are $SU(2)$, $U(1)$, and Z_2 respectively. We note that $SU(2)$, $U(1)$, and Z_2 happen to be the gauge groups that describe the low energy gauge fluctuations in the three spin liquids. This relation is not an accident. In general the gauge group of the low energy gauge fluctuations for a spin liquid described by an ansatz \bar{u}_{ij} is given by the IGG of the ansatz [16]. This result generalizes the analysis of the low energy gauge group based on the $SU(2)$ flux.

If an ansatz is invariant under the translation T_x followed by a gauge transformation G_x , then it is also invariant under the translation T_x followed by another gauge transformation G_0G_x , as long as $G_0 \in IGG$. So the gauge transformation associated with a symmetry transformation is not unique. The number of the choices of the gauge transformations is the number of the elements in IGG. We see that as sets, $PSG = SG \times IGG$ where SG is the symmetry group. But as groups, PSG is not the direct product of SG and IGG . It is a “twisted” product. Using the more rigorous mathematical notation, we have

$$SG = PSG/IGG$$

We may also say that the PSG is a projective extension of SG by IGG .

The $SU(2)$, $U(1)$ and Z_2 ansatz all have the same symmetry and hence the same symmetry group SG . They have different PSG 's since the same SG is extended by different IGG 's. Here we would like to remark that even for a given pair of SG and IGG , there are many different ways to extend the SG by the IGG , leading to many different PSG 's. For example there are over 100 ways to extend the symmetry group of a square lattice by a Z_2 IGG. This implies that there are over 100 different Z_2 spin liquids on a square lattice and those spin liquids all have the exactly the same symmetry! Finding different ways of extending a symmetry group SG is a pure mathematical problem. Such a calculation will lead to a (partial) classification of the quantum orders (and the spin liquids).

5. PSG is a universal property which protects gapless excitations

From the above discussion, we see that a PSG contains two parts. The first part is SG which describe the symmetry of the spin liquid. The second part is IGG which describe the gauge “symmetry” of the spin liquid. A generic elements in the PSG is a combination of the symmetry transformation and the gauge transformation.

We know that symmetry and gauge “symmetry” are universal properties, *ie* perturbative fluctuations cannot break the symmetry, nor can they break the gauge “symmetry”. So both SG and IGG are universal properties. This strongly suggests that the PSG is also a universal property.

To directly show a PSG to be a universal property, we note that the fermion mean-field Hamiltonian H_{mean} in Eq. (3) is invariant under the lattice symmetry and the $SU(2)$ gauge transformations Eq. (4). But the mean-field ansatz \bar{u}_{ij} is not invariant under the separate lattice symmetry and $SU(2)$ gauge transformations. So the mean-field state break the separate lattice symmetry and $SU(2)$ gauge “symmetry” down to a smaller symmetry. The symmetry group of this smaller symmetry is the PSG. So, the PSG is the symmetry of the mean-field theory with \bar{u}_{ij} ansatz. As a result, the PSG is the symmetry for the effective Lagrangian $\mathcal{L}_{\bar{u}_{ij}}(\psi, \delta u_{ij})$ that describes the fluctuations around the mean-field ansatz. If the mean-field fluctuations do not have any infrared divergence, then those fluctuations will be perturbative in nature and cannot change the symmetry – the PSG.

What do we mean by “perturbative fluctuations cannot change the PSG?” We know that a mean-field ground state is characterized by \bar{u}_{ij} . If we include perturbative fluctuations to improve our calculation of the mean-field energy $\langle \Phi_{spin}^{u_{ij}} | H | \Phi_{spin}^{u_{ij}} \rangle$, then we expect the \bar{u}_{ij} that minimize the improved mean-field energy to receive perturbative corrections $\delta \bar{u}_{ij}$. The statement “perturbative fluctuations cannot change the PSG?” means that \bar{u}_{ij} and $\bar{u}_{ij} + \delta \bar{u}_{ij}$ have the same PSG.

As the perturbative fluctuations (by definition) do not

change the phase, \bar{u}_{ij} and $\bar{u}_{ij} + \delta\bar{u}_{ij}$ describe the same phase. In other words, we can group \bar{u}_{ij} into classes (which are called universality classes) such that the \bar{u}_{ij} in each class are connected by the perturbative fluctuations. Each universality class describes one phase. We see that, if the above argument is true, then the ansatz in a universality class all share the same PSG. In other words, the universality classes (or the phases) are classified by the PSG's. Thus the PSG is a universal property. We can use the PSG to describe the quantum order in the spin liquid, as long as the low energy effective theory $\mathcal{L}_{\bar{u}_{ij}}(\psi, \delta u_{ij})$ does not have any infrared divergence.

In the standard renormalization group analysis of the stability of a phase or a critical point, one needs to include all the counter terms that have the right symmetries into the effective Lagrangian, since those terms can be generated by perturbative fluctuations. Then we examine if those allowed counter terms are relevant perturbations or not. In our problem, $\delta\bar{u}_{ij}$ discussed above correspond to those counter terms. The effective Lagrangian with the counter term is given by $\mathcal{L}_{\bar{u}_{ij}+\delta\bar{u}_{ij}}(\psi, \delta u_{ij})$. The new feature here is that it is incorrect to use the symmetry group alone to determine the allowed counter terms $\delta\bar{u}_{ij}$. We should use PSG to determine the allowed counter terms in our analysis of the stability of phases and critical points. That is the counter terms $\delta\bar{u}_{ij}$ should not change the PSG of \bar{u}_{ij} .

The Z_2 spin liquid Eq. (10) (and other 100 plus Z_2 spin liquids) contains no diverging fluctuations. So the PSG description of the quantum order is valid for this case. For the $SU(2)$ spin liquid Eq. (6) and the $U(1)$ spin liquid Eq. (8), their low energy effective theory Eq. (7) and Eq. (9) contain log divergence. These are marginal cases where the PSG description of the quantum order still apply. In a renormalization group analysis of the stability of the $U(1)$ spin liquid Eq. (8), one can show that, in a large N limit, the counter terms allowed by the $U(1)$ PSG Eq. (12) are all irrelevant [62, 63], even if we include the instanton effect [64]. Thus the (large N) $U(1)$ spin liquid is a stable quantum phase. One can also show that non of the allowed counter terms can give the gapless fermions and gapless gauge bosons an energy gap [16, 31]. Thus the gapless excitations in the $U(1)$ spin liquid are protected by the $U(1)$ PSG despite those gapless excitations interact down to zero energy.

E. An intuitive understand of quantum order and the emergent gauge bosons and fermions

The projective construction produces a correlated many-body ground state $|\Phi_{spin}\rangle = \mathcal{P}|\Psi_{mean}^{\bar{u}_{ij}}\rangle$. We may view the complicated correlation in the ground state as a pattern of quantum entanglement. The quantum order and the associated PSG is a characterization of such a pattern of entanglement. The gauge fluctuation above the many-body ground state can be viewed as a fluctuation of the entanglement. The fermion excitations can be

viewed as topological defects in the entanglement. From this point of view, the theory of quantum order can be regarded as a theory of many-body quantum entanglement.

V. AN EXACT SOLUBLE MODEL FROM PROJECTIVE CONSTRUCTION

Usually, the projective construction does not give us exact results. In this section, we are going to construct an exactly soluble model on 2D square lattice [15, 68]. The model has a property that the projective construction give us exact ground states and all other exact excited states.

A. An exact soluble model for the ψ -fermions

First, we would like to construct an exact soluble model for the ψ -fermions. It is convenient to write the exact soluble Hamiltonian in terms of four Majorana fermions

$$2\psi_{1,i} = \lambda_i^x + i\lambda_i^{\bar{x}}, \quad 2\psi_{2,i} = \lambda_i^y + i\lambda_i^{\bar{y}} \quad (14)$$

The Majorana fermions satisfy the algebra $\{\lambda_{a,i}, \lambda_{b,j}\} = 2\delta_{ab}\delta_{ij}$, where $a, b = x, \bar{x}, y, \bar{y}$. The exact soluble fermion Hamiltonian is given by

$$H = - \sum_i g\hat{F}_i, \quad (15)$$

$$\hat{F}_i = \hat{U}_{i,i+\hat{x}}\hat{U}_{i+\hat{x},i+\hat{x}+\hat{y}}\hat{U}_{i+\hat{x}+\hat{y},i+\hat{y}}\hat{U}_{i+\hat{y},i}, \\ \hat{U}_{i,i+\hat{x}} = \lambda_i^x\lambda_{i+\hat{x}}^{\bar{x}}, \quad \hat{U}_{i,i+\hat{y}} = \lambda_i^y\lambda_{i+\hat{y}}^{\bar{y}}, \quad \hat{U}_{ij} = -\hat{U}_{ji}.$$

To see why the above interacting fermion model is exactly soluble, we note that \hat{U}_{ij} commute with each other and H commute with all the \hat{U}_{ij} . So we can find the eigenvalues and eigenstates of H by finding the common eigenstates of \hat{U}_{ij} :

$$\hat{U}_{ij}|\{s_{ij}\}\rangle = s_{ij}|\{s_{ij}\}\rangle$$

Since $(\hat{U}_{ij})^2 = -1$ and $\hat{U}_{ij} = -\hat{U}_{ji}$, s_{ij} satisfies $s_{ij} = \pm i$ and $s_{ij} = -s_{ji}$. Since H is a function of \hat{U}_{ij} 's, $|\{s_{ij}\}\rangle$ is also an energy eigenstate of Eq. (15) with energy

$$E = - \sum_i gF_i, \\ F_i = s_{i,i+\hat{x}}s_{i+\hat{x},i+\hat{x}+\hat{y}}s_{i+\hat{x}+\hat{y},i+\hat{y}}s_{i+\hat{y},i}. \quad (16)$$

To see if $|\{s_{ij}\}\rangle$'s represent all the exact eigenstates of H , we need count the states. Let us assume the 2D square lattice to have N_{site} lattice sites and a periodic boundary condition in both directions. In this case the lattice has $2N_{site}$ links. Since there are total of $2^{2N_{site}}$ different choices of s_{ij} (two choices for each link), the states $|\{s_{ij}\}\rangle$ exhaust all the $4^{N_{site}}$ states in the (ψ_1, ψ_2)

Hilbert space. Thus the common eigenstates of \hat{U}_{ij} is not degenerate and the above approach allows us to obtain all the eigenstates and eigenvalues of the H .

We note that the eigenstate $|\{s_{ij}\}\rangle$ is the ground state of the following free fermion Hamiltonian

$$H_{mean} = \sum_{\langle ij \rangle} s_{ij} \hat{U}_{ij} \quad (17)$$

by choosing different $s_{ij} = \pm i$, the ground state of the above mean-field Hamiltonian give rise to all the eigenstates of the interacting fermion Hamiltonian Eq. (15).

B. An exact soluble spin-1/2 model

We note that the Hamiltonian H can only change the fermion number on each site, $n_i = \psi_{1,i}^\dagger \psi_{1,i} + \psi_{2,i}^\dagger \psi_{2,i}$, by an even number. Thus the H acts within a subspace which has an even number of fermions on each site. We will call such a subspace physical Hilbert space. The physical Hilbert space has only two states per site corresponding to a spin-up and a spin-down state. When restricted within the physical space, H actually describes a spin-1/2 system. To obtain the corresponding spin-1/2 Hamiltonian, we note that

$$\sigma_i^x = i\lambda_i^y \lambda_i^x, \quad \sigma_i^y = i\lambda_i^x \lambda_i^y, \quad \sigma_i^z = i\lambda_i^x \lambda_i^{\bar{x}} \quad (18)$$

act within the physical Hilbert space and satisfy the algebra of Pauli matrices. Thus we can identify σ_i^l as the spin operator. Using the fact that

$$(-)^{n_i} = \lambda_i^x \lambda_i^y \lambda_i^{\bar{x}} \lambda_i^{\bar{y}} = 1$$

within the physical Hilbert space, we can show that the fermion Hamiltonian Eq. (15) becomes (see Fig. 7)

$$H_{spin} = - \sum_i g \hat{F}_i, \quad \hat{F}_i = \sigma_i^x \sigma_{i+\hat{x}}^y \sigma_{i+\hat{x}+\hat{y}}^x \sigma_{i+\hat{y}}^y \quad (19)$$

within the physical Hilbert space.

C. The projective construction leads to exact results

All the states in the physical Hilbert space (*ie* all the states in the spin-1/2 model) can be obtained from the $|\{s_{ij}\}\rangle$ states by projecting into the physical Hilbert space: $\mathcal{P}|\{s_{ij}\}\rangle$. The projection operator is given by

$$\mathcal{P} = \prod_i \frac{1 + (-)^{n_i}}{2}$$

Since $[\mathcal{P}, H] = 0$, the projected state $\mathcal{P}|\{s_{ij}\}\rangle$, if non-zero, remain to be an eigenstate of H (or H_{spin}) and remain to have the same eigenvalue. We see that the ground state of the mean-field Hamiltonian Eq. (17), after the projection, give rise to all the exact eigenstates of the spin Hamiltonian Eq. (19). The project construction is exact for Eq. (19)!

D. The Z_2 gauge structure

The physical states (with even numbers of fermions per site) are invariant under local Z_2 transformations generated by

$$\hat{G} = \prod_i G_i^{n_i}$$

where G_i is an arbitrary function with only two values ± 1 . The Z_2 transformation change ψ_{ai} to $\tilde{\psi}_{ai} = G_i \psi_{ai}$ and s_{ij} to $\tilde{s}_{ij} = G_i s_{ij} G_j$, or more precisely

$$|\{\tilde{s}_{ij}\}\rangle = \hat{G}|\{s_{ij}\}\rangle$$

Using $\mathcal{P}\hat{G} = \mathcal{P}$, we find that $|\{s_{ij}\}\rangle$ and $|\{\tilde{s}_{ij}\}\rangle$ give rise to the same physical state after projection (if their projection is not zero):

$$\mathcal{P}|\{\tilde{s}_{ij}\}\rangle = \mathcal{P}|\{s_{ij}\}\rangle \quad (20)$$

Thus, s_{ij} is a many-to-one label of the physical spin state.

The above results indicate that we can view is_{ij} as a Z_2 gauge potential and the local Z_2 transformation is a Z_2 gauge transformation. Eq. (20) implies that gauge equivalent gauge potential described the same physical state. The fluctuations of s_{ij} is described by a Z_2 gauge theory, which is the low energy effective theory of the spin system Eq. (19).

VI. CLOSED-STRING CONDENSATION

The ground state of the exactly soluble model contains a special property – closed-string condensation. In this section, we will see that the closed-string condensation is intimately related to the emergence of the gauge structure and the fermions. We have shown that the ground state of the exactly soluble model can be constructed via the projective construction. This indicates that the projective construction is probably just a trick to construct string condensed states.

A. String operators and closed-string condensation

First let us define a string C as a curve that connect the midpoints of neighboring links (see Fig. 7). The string operator has the following form

$$W(C) = \prod_n \sigma_{i_n}^{l_n} \quad (21)$$

where i_n are sites on the string. $l_n = z$ if the string does not turn at site i_n . $l_n = x$ or y if the string makes a turn at site i_n . $l_n = x$ if the turn forms a upper-right or lower-left corner. $l_n = y$ if the turn forms a lower-right or upper-left corner. (See Fig. 7.)

By an explicit calculation, one can show that the closed string operator defined above commute with the spin

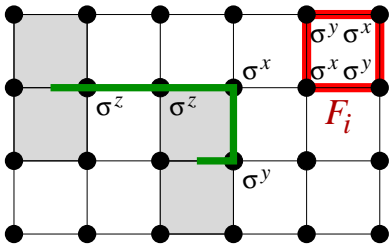


FIG. 7: The string is formed by a curve connecting the midpoints of the neighboring links. The string operator is formed by the product of $\sigma_i^{x,y,z}$ for sites on the string. The operator \hat{F}_i is also presented.

Hamiltonian Eq. (19). So the ground state $|\text{ground}\rangle$ of the spin Hamiltonian is also an eigenstate of the closed-string operator. Since the eigenvalues of the closed string operator are ± 1 . We have $\langle \text{ground} | W(C_{\text{closed}}) | \text{ground} \rangle = \pm 1$. The ground state has a closed-string condensation.

We note that in a symmetry breaking state, the operator representing the order parameter has a non-zero expectation value $\langle \hat{O}_i \rangle = \phi$. If we define a string operator as the product of $\frac{\hat{O}_i}{\phi}$ along a loop, the average of the string operator will be non-zero. However, we do not regard the non-zero average of such a string operator to represent a string condensation. The real closed-string condensation must be “unbreakable”, in the sense that we cannot break a closed string operator into several segments and find condensation in each segment. The string operator $\prod_i \frac{\hat{O}_i}{\phi}$ does not satisfy this property. However, the string operator defined in Eq. (21) is indeed unbreakable. This is because an open-string operator does not commute with the spin Hamiltonian and does not condense $\langle \text{ground} | W(C_{\text{open}}) | \text{ground} \rangle = 0$. It is such a “unbreakable” closed-string condensation indicates a new order in the ground state.

In terms of the Majorana fermions, the closed-string can be written as

$$W(C_{\text{closed}}) = \prod_{\langle ij \rangle} i \hat{U}_{ij}$$

where $\langle ij \rangle$ are the nearest neighbor links that form the closed string C_{closed} . For a spin state obtained from the ansatz s_{ij} (via the projection, $\mathcal{P}|\{s_{ij}\}\rangle$), we find that

$$\langle \{s_{ij}\} | \mathcal{P} W(C_{\text{closed}}) \mathcal{P} | \{s_{ij}\} \rangle = \prod_{\langle ij \rangle} i s_{ij}$$

(note that for closed strings $[W(C_{\text{closed}}), \mathcal{P}] = 0$). Therefore the closed string operator is nothing but the Wegner-Wilson loop operator [22, 69] for the corresponding Z_2 gauge theory. We see an intimate relation between the closed-string condensation and the emergence of a gauge structure.

B. Open string operators and Z_2 charges

If the fluctuations of s_{ij} represent Z_2 gauge fluctuations, what are the Z_2 charges? In a gauge theory, we know that a Wegner-Wilson operator for an open string is not gauge invariant and is not a physical operator (*ie* is not an operator that acts within the physical Hilbert space). The open-string operator defined in Eq. (21) act with the physical Hilbert space and is a gauge invariant physical operator. Thus the Wegner-Wilson operator for an open string does not correspond to our open string operator. However, in the gauge theory, two charge operators connected by the Wegner-Wilson operator, $\phi_{\mathbf{x}_1}^\dagger \phi_{\mathbf{x}_2} e^{i \int_{\mathbf{x}_1}^{\mathbf{x}_2} dx \cdot \mathbf{a}}$, is gauge invariant. It is such an operator that correspond to our open string operator.

In the gauge theory, the Wegner-Wilson operator create a loop of electric flux. The closed-string operator defined in Eq. (21) has the same physical meaning. In the gauge theory, the operator $\phi_{\mathbf{x}_1}^\dagger \phi_{\mathbf{x}_2} e^{i \int_{\mathbf{x}_1}^{\mathbf{x}_2} dx \cdot \mathbf{a}}$ creates two charges connected by a line of the electric flux. Our open string operator defined in Eq. (21) does the same thing: it creates two Z_2 charges at its two ends and an electric flux line connecting the two charges.

Due to the closed-string condensation, the string connecting the two Z_2 charges is unobservable and costs no energy. Thus the open string does not create an extended line-like object, it creates two point-like objects at its ends. Those point-like objects are the Z_2 charges. Despite the point-like appearance, the Z_2 charges are intrinsically non-local. There is no way to create a lone Z_2 charge. Because of this, the Z_2 charge (and other gauge charges) usually carry fractional quantum numbers.

C. Statistics of the Z_2 charges

What is the statistics of the Z_2 charges? Usually, bosons are defined as particles described by commuting operators and fermions as particles described by anti-commuting operators. But this definition is too formal and is hard to apply to our case. To find a new way to calculate statistics, we need to gain a more physical understanding of the difference between bosons and fermions.

Let us consider the following many-body hopping system. The Hilbert space is formed by a zero-particle state $|0\rangle$, one-particle states $|\mathbf{i}_1\rangle$, two-particle states $|\mathbf{i}_1, \mathbf{i}_2\rangle$, etc, where \mathbf{i}_n labels the sites in a lattice. As an identical particle system, the state $|\mathbf{i}_1, \mathbf{i}_2, \dots\rangle$ does not depend on the order of the indexes $\mathbf{i}_1, \mathbf{i}_2, \dots$. For example $|\mathbf{i}_1, \mathbf{i}_2\rangle = |\mathbf{i}_2, \mathbf{i}_1\rangle$. There are no doubly-occupied sites and we assume $|\mathbf{i}_1, \mathbf{i}_2, \dots\rangle = 0$ if $\mathbf{i}_m = \mathbf{i}_n$.

A hopping operator \hat{t}_{ij} is defined as follows. When \hat{t}_{ij} acts on state $|\mathbf{i}_1, \mathbf{i}_2, \dots\rangle$, if there is a particle at site \mathbf{j} but no particle at site \mathbf{i} , then \hat{t}_{ij} moves the particle at site \mathbf{j} to site \mathbf{i} and multiplies a complex amplitude $t(\mathbf{i}, \mathbf{j}; \mathbf{i}_1, \mathbf{i}_2, \dots)$ to the resulting state. Note that the amplitude may de-

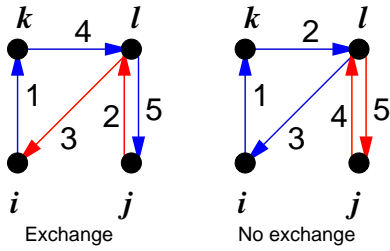


FIG. 8: (a) The first way to arrange the five hops swaps the two particles. (b) The second way to arrange the same five hops does not swap the two particles.

pend on the locations of all the other particles. The Hamiltonian of our system is given by

$$H_{hop} = \sum_{\langle ij \rangle} \hat{t}_{ij}$$

where the sum $\sum_{\langle ij \rangle}$ is over a certain set of pairs $\langle ij \rangle$, such as nearest-neighbor pairs. In order for the above Hamiltonian to represent a local system we require that

$$[\hat{t}_{ij}, \hat{t}_{kl}] = 0$$

if i, j, k , and l are all different.

What does the hopping Hamiltonian H_{hop} describe? A hard-core boson system or a fermion system? Whether a many-body hopping system is a boson system or a fermion system (or even some other statistical systems) has nothing to do with the Hilbert space. The fact that the many-body states are labeled by symmetric indexes (eg $|\mathbf{i}_1, \mathbf{i}_2\rangle = |\mathbf{i}_2, \mathbf{i}_1\rangle$) does not imply that the many-body system is a boson system. The statistics are determined by the Hamiltonian H_{hop} .

Clearly, when the hopping amplitude $t(i, j; \mathbf{i}_1, \mathbf{i}_2, \dots)$ only depends on i and j , $t(i, j; \mathbf{i}_1, \mathbf{i}_2, \dots) = t(i, j)$, the many-body hopping Hamiltonian will describe a hard-core boson system. The issue is under what condition the many-body hopping Hamiltonian describes a fermion system.

This problem was solved in Ref. [19]. It was found that the many-body hopping Hamiltonian describes a fermion system if the hopping operators satisfy

$$\hat{t}_{lk}\hat{t}_{il}\hat{t}_{lj} = -\hat{t}_{lj}\hat{t}_{il}\hat{t}_{lk} \quad (22)$$

for any three hopping operators \hat{t}_{lj} , \hat{t}_{il} , and \hat{t}_{lk} with i, j, k , and l all being different. (Note that the algebra has a structure $\hat{t}_1\hat{t}_2\hat{t}_3 = -\hat{t}_3\hat{t}_2\hat{t}_1$.)

To understand this result, consider the state $|\mathbf{i}, \mathbf{j}, \dots\rangle$ with two particles at i, j , and possibly other particles further away. We apply a set of five hopping operators $\{\hat{t}_{jl}, \hat{t}_{lk}, \hat{t}_{il}, \hat{t}_{lj}, \hat{t}_{ki}\}$ to the state $|\mathbf{i}, \mathbf{j}, \dots\rangle$ but with different order (set Fig. 8)

$$\begin{aligned} \hat{t}_{jl}\hat{t}_{lk}\hat{t}_{il}\hat{t}_{lj}\hat{t}_{ki}|\mathbf{i}, \mathbf{j}, \dots\rangle &= C_1|\mathbf{i}, \mathbf{j}, \dots\rangle \\ \hat{t}_{jl}\hat{t}_{lj}\hat{t}_{il}\hat{t}_{lk}\hat{t}_{ki}|\mathbf{i}, \mathbf{j}, \dots\rangle &= C_2|\mathbf{i}, \mathbf{j}, \dots\rangle \end{aligned}$$

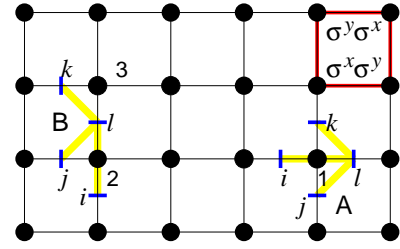


FIG. 9: The Z_2 charge live on the links. The hopping of the Z_2 charges is induced by the open string operator.

where we have assumed that there are no particles at sites k and l . We note that after five hops we get back to the original state $|\mathbf{i}, \mathbf{j}, \dots\rangle$ with additional phases $C_{1,2}$. However, from Fig. 8, we see that the first way to arrange the five hops (Fig. 8a) swaps the two particles at i and j , while the second way (Fig. 8b) does not swap the two particles. Since the two hopping schemes use the same set of five hops, the difference between C_1 and C_2 is due to exchanging the two particles. Thus we require $C_1 = -C_2$ in order for the many-body hopping Hamiltonian to describe a fermion system. Noting that the first and the last hops are the same in the two hopping schemes, we find that $C_1 = -C_2$ if the hopping operators satisfy Eq. (22). Eq. (22) serves as an alternative definition of Fermi statistics if we do not want to use anti-commuting algebra.

For our spin model Eq. (19), the open strings end at the midpoint of the links. So the Z_2 charges live on the links. To apply the above result to the Z_2 charges, we note that an open string operator connecting midpoints i and j (see Fig. 9) play the role of a hopping operator \hat{t}_{ij} . Near the site 1 in Fig. 9, the hopping operators between the midpoints i, j, k , and l are given by

$$\hat{t}_{lj} = \sigma_1^y, \quad \hat{t}_{il} = \sigma_1^z, \quad \hat{t}_{lk} = \sigma_1^x.$$

The fermion hopping algebra Eq. (22) becomes $\sigma_1^y \sigma_1^z \sigma_1^x = -\sigma_1^x \sigma_1^z \sigma_1^y$ which is satisfied. Near the site 2 and 3 in Fig. 9, the hopping operators between the midpoints i, j, k , and l are given by

$$\hat{t}_{lj} = \sigma_2^y, \quad \hat{t}_{il} = \sigma_2^z, \quad \hat{t}_{lk} = \sigma_3^x$$

The fermion hopping algebra becomes $\sigma_2^y \sigma_2^z \sigma_3^x = -\sigma_3^x \sigma_2^z \sigma_2^y$ which is again satisfied. We see that the hopping operators of the Z_2 charges satisfy a fermion hopping algebra. So the Z_2 charges are fermions. Fermions can emerge in a pure bosonic model as ends of condensed strings.

VII. SUMMARY

Symmetry breaking have dominated our understanding of phase and phase transition for over 50 years. We now know that symmetry breaking cannot describe all the possible orders that matter can have. The study of

the quantum order [16] and the associated condensation of string-nets [24, 39] and other extended objects suggest that a new world beyond symmetry breaking exists. It appears that the new world of quantum order is much richer than the world of symmetry breaking order. The most striking picture from the new world is that gauge bosons, fermions, and string-net condensation are just the different sides of the same coin. Or in other words, the string-net condensation provides a way to unify gauge interactions and Fermi statistics! So far, we have only seen some small fragmented pieces of the new world. The exciting time is still ahead of us. Comparing with our understanding of symmetry breaking order, we need to understand the following aspects of quantum order and the associated string-net condensation:

(1) We know that the group theory is the mathematical frame work behind the symmetry breaking order. What is the mathematical frame work behind string condensa-

tion? PSG only provides a partial answer to this question. A recent work [24] suggests that tensor category theory may play the same role in string-net condensed states as group theory in symmetry breaking states.

(2) We know that crystal orders can be measured through X-ray diffraction. How to measure different quantum orders associated with different string-net condensations?

(3) We know that many material contain non-trivial symmetry breaking orders. What material has string-net condensation and emergent gauge bosons and fermions? (A believer can always say that we actually live inside one such material. However, one needs more to convince a non-believer.)

The more we explore, the more are we fascinated by the endless richness that the nature reveals to us.

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